Mikailalsys Journal of
Advanced Engineering International

e-ISSN : 3046-9694
p-ISSN : 3046-8914

Index : Harvard, Boston, Sydney University, Dimensions, Lens, Scilit, Semantic, Google, etc

https://doi.org/10.58578/MJAEI.v2i3.6807

The Concept of Differentiation and Integration of Paraletrix: A Generalization of Rhotrix

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Article Info:

Submitted:	**Revised:**	**Accepted:**	**Published:**
Jul 23, 2025	Aug 19, 2025	Aug 31, 2025	Sep 5, 2025

Abstract

The concepts of matrix-tertions and matrix-noittrets were first introduced by Atanssov, K. T. and Shannon, A. G. [1] as mathematical enrichment exercises involving objects that lie between two-dimensional vectors and 2×2 matrices. This idea was later extended by Ajibade, A. O. [2] through the introduction of rhotrices, mathematical structures positioned between 2×2 and 3×3 matrices. Various multiplication operations for rhotrices, including heart-oriented and row–column multiplications, have been studied extensively, yielding several important results. Building on these developments, the paraletrix was introduced as a generalization of the rhotrix, defined as a structure in which the number of rows and columns need not be equal. In this paper, we extend the theory of paraletrices by introducing the concepts of differentiation and integration with respect to an independent variable occurring in a function, thereby contributing to the broader mathematical framework of generalized matrix-like objects.

Keywords: Paraletrix; Rhotrix; Differentiation; Integration; Heart of Paraletrix

Volume 2, Issue 3, 2025; Pages 350-358
https://ejournal.yasin-alsys.org/MJAEI

Introduction

The idea of rhotrix as an object that has its elements arranged in a rhomboidal nature was first introduced by Ajibade [2] as an extension of the initiative on matrix-tertions and matrix-noitrets of Atanassov and Shannon [1]. Matrices are either square or rectangular depending on the number of rows and columns while rhotrices always have the same number of rows and columns. The binary operations of addition (+) and multiplication ($\circ$) over rhotrices are defined by Ajibade [2] as follows: Let R and Q be two rhotrices defined as

$$R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \text{ and } Q = \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle . \tag{1}$$

Then the binary operation of addition (+) is given by:

$$R + Q = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & a+f & \\ b+g & h(R)+h(Q) & d+j \\ & e+k & \end{array} \right\rangle$$

and multiplication ($\circ$) is

$$R \circ Q = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & ah(Q)+fh(R) & \\ bh(Q)+gh(R) & h(R)h(Q) & dh(Q)+jh(R) \\ & eh(Q)+kh(R) & \end{array} \right\rangle$$

Another multiplication method for rhotrices called *row-column multiplication* was introduced by Sani [3] to answer some questions raised by Ajibade [2]. The row-column multiplication method is in a similar way as that of multiplication of matrices and is illustrated using the rhotrices R and Q defined above in (1) as

$$R \circ Q = \left\langle \begin{array}{ccc} & af+dg & \\ bf+eg & h(R)h(Q) & aj+dk \\ & bj+ek & \end{array} \right\rangle .$$

A generalization of the row-column multiplication method for *n*-dimensional rhotrices was given by Sani [4]. That is, given *n*-dimensional rhotrices $R_n = \langle a_{ij}, c_{lk} \rangle$ and $Q_n = \langle b_{ij}, d_{lk} \rangle$ the multiplication of R_n and Q_n is as follows:

$$R_n \circ Q_n = \langle a_{i_1 j_1}, c_{l_1 k_1} \rangle \circ \langle b_{i_2 j_2}, d_{l_2 k_2} \rangle = \left\langle \sum_{i_2 j_1 = 1}^{t} (a_{i_1 j_1} b_{i_2 j_2}), \sum_{l_2 k_1 = 1}^{t-1} (c_{l_1 k_1} d_{l_2 k_2}) \right\rangle, \ t = (n+1)/2 . \ (2)$$

Rhotrix vectors (either row vectors or column vectors) can be represented in *t* different ways where $t = (n+1)/2$. This is different compared to vectors in matrices that can be represented in a unique way.

Matrices are either square or rectangular depending on the number of rows and columns while rhotrices always have the same number of rows and columns. Having this observation as a motivation, we introduced a structure called paraletrix as an object that has its elements arranged in a parallelogram whose number of rows and columns are not necessarily the same. To the best of our knowledge paraletrix has not received attention in the past and no work has been done on differentiation and integration of paraletrix. It is the main aim of this paper to present the concept of differentiation and integration of paraletrix, a generalization of rhotrix.

Materials and Methods

Paraletrix and its basic Properties.

I. Let m *and* n be the number of rows and columns of an arbitrary paraletrix where $m, n \in \{2k+1 : k \in \mathrm{N}\}$.

An $m \times n$ –dimensional paraletrix is of the form: $P_{m \times n} = \langle a_{ij,} c_{lk} \rangle$

$$= \left\langle \begin{array}{ccccccccc}
 & & & & a_{11} & & & & \\
 & & & a_{21} & c_{11} & a_{12} & & & \\
 & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
a_{\frac{m+1}{2}1} & c_{\frac{m-1}{2}1} & \cdots & \cdots & h(p) & \cdots & \cdots & c_{1\frac{n-1}{2}} & a_{1\frac{n+1}{2}} \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & a_{\frac{m+1}{2}\frac{n-3}{2}} & c_{\frac{m-1}{2}\frac{n-3}{2}} & a_{\frac{m-1}{2}\frac{n-1}{2}} & c_{\frac{m-3}{2}\frac{n-1}{2}} & a_{\frac{m-3}{2}\frac{n+1}{2}} & & \\
 & & & a_{\frac{m+1}{2}\frac{n-1}{2}} & c_{\frac{m-1}{2}\frac{n-1}{2}} & a_{\frac{m-1}{2}\frac{n+1}{2}} & & & \\
 & & & & a_{\frac{m+1}{2}\frac{n+1}{2}} & & & &
\end{array} \right\rangle \tag{3}$$

Where $a_{ij}, c_{ik} \in R$ for

$$i = 1,2,\ldots,\frac{m+1}{2}, j = 1,2,\ldots,\frac{n+1}{2} \; and \quad i = 1,2,\ldots,\frac{m-1}{2}, k = 1,2,\ldots,\frac{n-1}{2}$$

For example,

$$P_{3\times5} = \left\langle \begin{array}{ccccc}
 & & a_{11} & & \\
 & a_{21} & c_{11} & a_{12} & \\
 & & a_{22} & c_{12} & a_{13} \\
 & & & a_{23} &
\end{array} \right\rangle \tag{4}$$

Equation (4) is a paraletrix with three rows:

$$\begin{array}{ccccccc}
a_{11} & & c_{11} & and & a_{21} & & \\
 & a_{12} & & c_{12} & & a_{22} & \\
 & & a_{13} & & & & a_{23}
\end{array}$$

and five columns:

$$\begin{array}{ccccccc}
 & a_{11} & & & a_{12} & & and \; a_{13} \\
a_{21} & & c_{11} & a_{22} & & c_{12} & a_{23}
\end{array}$$

It is important to mention here that the name paraletrix is as a result of the parallelogram nature of the arrangement of its entries. From the paraletrix defined in (3), we can extract two matrices:

$A = (a_{ij})$ *and* $C = (c_{lj})$ where $i = 1, 2, . . . , \frac{m+1}{2}$, $j = 1, 2, . . . , \frac{n+1}{2}$, and $k = 1, 2, . . . , \frac{m-1}{2}$, $l = 1, 2, . . . , \frac{n-1}{2}$. Matrix $A = (a_{ij})$ is called the major matrix and $C = (c_{lj})$ is the minor matrix and are either square or rectangular depending on the dimension of the paraletrix.

The Heart of a Paraletrix

The Heart (or centre) of a paraletrix is simply defined as the element in a paraletrix which divides the entries into two equal parts. In other words, the heart of an $m \times n$ -dimensional paraletrix $P_{m\times n}$ is the element located at the $\frac{1}{2}[\frac{1}{2}((m \times n) + 1) + 1]$ position of the paraletrix. It worthy to mention here that not every paraletrix has a heart, A. Aminu and O. Michael [5], and this is addressed in the following lemma.

Lemma 1.

Let $P_{m\times n}$ be an $m \times n$ –dimensional paraletrix. If $P_{m\times n}$ has a heart then the heart is unique.

Proof. By contradiction, suppose h_1 and h_2 are hearts of $P_{m\times n}$ such that $h_1 \neq h_2$. It follows from the definition of heart that h_1 and h_2 lies in the $\frac{1}{2}[\frac{1}{2}((m \times n) + 1) + 1]$ position of $P_{m\times n}$. Since the position is unique then $h_1 = h_2$ and the statement follows.

Paraletrix Differentiation

If the elements of a paraletrix $P_{m\times n}$ are functions of a variable x, then the paraletrix is called a paraletrix function of x; that is:

$P_{m\times n} = P_{m\times n}(x) = \left\langle a_{ij}(x), c_{lk}(x) \right\rangle$ and the differential coefficients of $P_{m\times n}$ with respect to x is

$$\frac{d}{dx} P_{m\times n}(x) = \left\langle \frac{d}{dx} a_{ij}(x), \frac{d}{dx} c_{lk}(x) \right\rangle$$

And the nth order derivative is given as:

$$\frac{d^n}{dx^n}P_{m\times n}(x)=\left\langle \frac{d^n}{dx^n}a_{ij}(x),\frac{d^n}{dx^n}c_{lk}(x)\right\rangle, \qquad for \qquad n=1,2,3,\ldots$$

Thus the elements of the differentiated paraletrix $\frac{d}{dx}P_{m\times n}(x)$ are the derivatives of the corresponding elements of $P_{m\times n}(x)$ of equation (3) .

Theorem 1. (Derivative of sum of two paraletrix)

Let $P_{m\times n}(x)\ and\ Q_{r\times s}(x)$ be two paraletrices, each with differentiable elements. Then

$$\frac{d}{dx}[P_{m\times n}(x)+Q_{r\times s}(x)]=\left\langle \frac{d}{dx}P_{m\times n}(x)+\frac{d}{dx}Q_{r\times s}(x)\right\rangle$$

Proof. The proof is obvious from the definition of sum of two paraletrices.

Theorem 2. (Derivative of the product of two paraletrix)

Let $P_{m\times n}(x)\ and\ Q_{r\times s}(x)$ be two paraletrices which are compatible for multiplication, each with differentiable elements. Then

$$\frac{d}{dx}[P_{m\times n}(x)Q_{r\times s}(x)]=\left\langle P_{m\times n}(x)\frac{d}{dx}Q_{r\times s}(x)+Q_{r\times s}(x)\frac{d}{dx}P_{m\times n}(x)\right\rangle$$

Proof. The proof follows immediately from the definition of the paraletrix multiplication.

Example 1. Find $\frac{d}{dx}P_{3\times 5}(x)$**,** given that

$$P_{3\times 5}(x)=\left\langle \begin{array}{cccc} & -x^3 & & \\ \frac{1}{x} & 1-x & 7x & \\ & \cos x & -5 & e^{3x} \\ & & \sin x & \end{array}\right\rangle$$

Solution

$$\frac{d}{dx}P_{3\times 5}(x)=\left\langle \begin{array}{cccc} & \frac{d}{dx}(-x^3) & & \\ \frac{d}{dx}\frac{1}{x} & \frac{d}{dx}(1-x) & \frac{d}{dx}7x & \\ & \frac{d}{dx}\cos x & \frac{d}{dx}(-5) & \frac{d}{dx}(e^{3x}) \\ & & \frac{d}{dx}\sin x & \end{array} \right\rangle$$

$$=\left\langle \begin{array}{cccc} & -3x^2 & & \\ -\frac{1}{x^2} & -1 & 7 & \\ & -\sin x & 0 & 3e^{3x} \\ & & \cos x & \end{array} \right\rangle$$

Paraletrix Integration

Just as we can find the derivative of a paraletrix $P_{m\times n}$ with respect to an independent variable x, we can as well find the anti-derivative or the definite integral of $P_{m\times n}$. The integral of $P_{m\times n}(x)$, either definite or indefinite, is obtained by integrating each element of $P_{m\times n}(x)$ as follows:

$$\int_a^b P_{m\times n}(x)dx=\left\langle [\int_a^b a_{ij}(x), c_{lk}(x)]dx \right\rangle=\left\langle \int_a^b a_{ij}(x)dx, \int_a^b c_{lk}(x)dx \right\rangle \text{ and}$$

$$\int P_{m\times n}(x)dx=\left\langle [\int a_{ij}(x), c_{lk}(x)]dx \right\rangle=\left\langle \int a_{ij}(x)dx, \int c_{lk}(x)dx \right\rangle$$

Thus we equally integrate equation (3) just as we did in differentiation.

Theorem 4.

Let $P_{m\times n}(x)\ and\ Q_{r\times s}(x)$ be two paraletrices. Then

$$\int P_{m\times n}(x)dx+\int Q_{r\times s}(x)dx=\left\langle \int P_{m\times n}(x)dx+\int Q_{r\times s}(x)dx \right\rangle$$

Proof. This follows immediately from the definition of sum of two paraletrices.

Example 3.

Find $\int P_{3\times 5}(x)$; where $\int P_{3\times 5}(x)$ is as defined above in example 1.

Solution.

$$\int P_{3\times 5}(x) = \int [\left\langle \begin{array}{cccc} & -x^3 & & \\ \frac{1}{x} & 1-x & 7x & \\ & \cos x & -5 & e^{3x} \\ & & \sin x & \end{array} \right\rangle]dx$$

$$= \left\langle \begin{array}{cccc} & \int(-x^3)dx & & \\ \int \frac{1}{x}dx & \int(1-x)dx & \int 7xdx & \\ & \int \cos xdx & \int(-5)dx & \int e^{3x}dx \\ & & \int \sin xdx & \end{array} \right\rangle$$

$$\left\langle \begin{array}{cccc} & -\frac{x^4}{4}+C_1 & & \\ \ln x + C_6 & x-\frac{x^2}{2}+C_4 & \frac{7x^2}{2}+C_2 & \\ & \sin x + C_7 & -5x+C_5 & \frac{e^{3x}}{3}+C_3 \\ & & -\cos x + C_8 & \end{array} \right\rangle$$

Conclusion

We have presented the concept of differentiation and integration of paraletrix, a generalization of rhotrix. It is worthy to mention that a paraletrix can only be added or multiplied together if they have the same dimension and also if the heart exist. To the best of our knowledge, paraletrix has not received attention in the past and the concept of differentiation and integration of paraletrix has not surface elsewhere in recent times. This paper is open to further investigation.

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