Mikailalsys Journal of
Advanced Engineering International

e-ISSN : 3046-9694
p-ISSN : 3046-8914

Index : Harvard, Boston, Sydney University, Dimensions, Lens, Scilit, Semantic, Google, etc

https://doi.org/10.58578/MJAEI.v2i1.4811

Generalized K-Fibonacci Sequence of Q- Matrix

Nand Kishor Kumar & Suresh Kumar Sahani
Tri-Chandra Campus, Tribhuvan University, Nepal
Rajarsi Janak University, Janapurdham, Nepal
nandkishorkumar2025@gmail.com; sureshsahani@rju.edu.np

Article Info:

Submitted:	Revised:	Accepted:	Published:
Dec 28, 2024	Jan 12, 2025	Jan 24, 2025	Jan 29, 2025

Abstract

This article explores the generalized K-Fibonacci sequence derived from matrices, extending the classical Fibonacci sequence to matrix representations. The properties and its applications are also described. The interplay of matrix algebra and generalized Fibonacci sequences offers insights into advanced sequence theory.

Keywords: Generalized K-Fibonacci Sequences, Q-Matrix, Binet Formula

Introduction

The Fibonacci sequence is a cornerstone of mathematical study with applications ranging from number theory to computer science. Its generalized versions, such as the K-Fibonacci sequence, allow for broader exploration of recurrence relations.

Numerous intriguing characteristics and uses of the Fibonacci numbers may be found in both science and the arts [1]. For their amazing qualities and uses, a variety of papers have been shown. One benefit of Fibonacci numbers is their generalizability [2-4]. The Fibonacci sequence may therefore be modified in a number of ways, and these generalized forms—such as matrix methods—have a number of intriguing characteristics. Numerous generalizations of these figures have been offered in different formats [5,6]. This is the most popular method for analyzing the Fibonacci sequence, and the so-called Fibonacci Q-matrix was recently defined [7, 8,17,18]:

$$Q_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ so that } Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \qquad (1)$$

where $F_{n+1} = F_n + F_{n-1}$ with the initial conditions; $F_1 = 1$ and $F_0 = 0$.

The 'Lucas Matrix,' introduced by Brenner, illustrates the initial application of the Q-matrix [15,16]. Charles H. King originally introduced the term "Q-matrix" in his 1960 master's thesis, "Some properties of the Fibonacci numbers," at San Jose State College [9]. Basin and Hoggatt [10] at least mention this source, and among Fibonacci enthusiasts, the idea took off like wildfire.

Following its quarterly publication by a number of distinguished authors, researchers, and academics, the Q-matrix technique emerged as a crucial instrument in the study of Fibonacci characteristics [14]. Gould [7] introduced the Q-matrix's history and higher dimensional difficulties. Using the matrix approach, Silvester et al. [11–13] recorded a few characteristics.

Definition 1. Let k and m be any positive real and fixed positive integer for $n \geq 2$, then the generalized k-Fibonacci sequence is given by

$Q_{k,n} = kQ_{k,n-1} + Q_{k,n-2}$ with initial conditions $Q_{k,0} = m, Q_{k,1} = km$

Two matrix representations of Q_1 and Q_2 for $Q_{k,n}$ can be defined as

$$Q_1 = \begin{bmatrix} k^2+1 & k \\ k & 1 \end{bmatrix} \text{ and } Q_2 = \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix} \qquad (2)$$

The generalized k-Fibonacci sequence for Q_1 matrix

Theorem 1. Let n be a positive integer, then Q_1 is defined as

$$Q_1^n = m^{-1} \begin{bmatrix} Q_{k,2n} & Q_{k,2n-1} \\ Q_{k,2n-1} & Q_{k,2n-2} \end{bmatrix} \qquad (3)$$

Proof. Let $n = 1$, we have

$$Q_1 = m^{-1}\begin{bmatrix} Q_{k,2} & Q_{k,1} \\ Q_{k,1} & Q_{k,0} \end{bmatrix} = \begin{bmatrix} k^2+1 & k \\ k & 1 \end{bmatrix}$$

Let this result is true for n, also we will show for $n + 1$ also.

$$\begin{aligned} Q_1^{n+1} &= Q_1^n Q_1 \\ &= m^{-1}\begin{bmatrix} Q_{k,2n} & Q_{k,2n-1} \\ Q_{k,2n-1} & Q_{k,2n-2} \end{bmatrix}\begin{bmatrix} k^2+1 & k \\ k & 1 \end{bmatrix} \\ &= m^{-1}\begin{bmatrix} k^2Q_{k,2n} + Q_{k,2n} + kQ_{k,2n-1} & kQ_{k,2n} + Q_{k,2n-1} \\ k^2Q_{k,2n-1} + Q_{k,2n-1} + kQ_{k,2n-2} & kQ_{k,2n-1} + Q_{k,2n-2} \end{bmatrix} \\ &= m^{-1}\begin{bmatrix} k(kQ_{k,2n} + Q_{k,2n-1}) + Q_{k,2n} & Q_{k,2n+1} \\ k(kQ_{k,2n-1} + Q_{k,2n-2}) + Q_{k,2n-1} & Q_{k,2n} \end{bmatrix} \\ &= m^{-1}\begin{bmatrix} Q_{k,2n+2} & Q_{k,2n+1} \\ Q_{k,2n+1} & Q_{k,2n} \end{bmatrix} \end{aligned}$$

Theorem 2. [16]

$$mQ_{k,2n} = Q_{k,n}^2 + Q_{k,n-1}^2, n \geq 1$$
$$mQ_{k,2n-1} = Q_{k,n}Q_{k,n-1} + Q_{k,n-1}Q_{k,n-2}, n \geq 2$$

$$\begin{aligned} Q_1^n &= Q^{\frac{n}{2}}\ Q^{\frac{n}{2}} \\ &= m^{-2}\begin{bmatrix} Q_{,k,n} & Q_{k,n-1} \\ Q_{k,n-1} & Q_{k,n-2} \end{bmatrix}\begin{bmatrix} Q_{k,n} & Q_{k,n-1} \\ Q_{k,n-1} & Q_{k,n-2} \end{bmatrix} \\ &= m^{-2}\begin{bmatrix} Q_{k,n}^2 + Q_{k,n-1}^2 & Q_{k,n}Q_{k,n-1} + Q_{k,n-1}Q_{k,n-2} \\ Q_{k,n}Q_{k,n-1} + Q_{k,n-2}Q_{k,n-1} & Q_{k,n}^2 + Q_{k,n-2}^2 \end{bmatrix}. \end{aligned}$$

Theorem 1. has defined,

$$Q_1^n = m^{-1}\begin{bmatrix} Q_{k,2n} & Q_{k,2n-1} \\ Q_{k,2n-1} & Q_{k,2n-2} \end{bmatrix}$$

On comparing these two matrices then

$$m^{-1}Q_{k,2n} = m^{-2}(Q_{k,n}^2 + Q_{k,n-1}^2)_{\text{n}}$$
$$mQ_{k,2n} = Q_{k,n}^2 + Q_{k,n-1}^2$$

and

$$m^{-1}Q_{k,2n-1} = m^{-2}(Q_{k,n}Q_{k,n-1} + Q_{k,n-2}Q_{k,n-1})$$
$$mQ_{k,2n-1} = Q_{k,n}Q_{k,n-1} + Q_{k,n-2}Q_{k,n-1}$$

Theorem 3. For the integers n and r,

$$mQ_{k,2n+2r} = Q_{k,2n}Q_{k,2r} + Q_{k,2n-1}Q_{k,2r-1}, n, r \geq 1$$
$$mQ_{k,2n+2r-1} = Q_{k,2n}Q_{k,2r-1} + Q_{k,2n-1}Q_{k,2r-2}, n, r \geq 1$$

Proof. Q_1^{n+r} has its summation form [16]

$$Q_1^{n+r} = Q_1^n Q_1^r$$
$$= m^{-2}\begin{bmatrix} Q_{k,2n} & Q_{k,2n-1} \\ Q_{k,2n-1} & Q_{k,2n-2} \end{bmatrix}\begin{bmatrix} Q_{k,2r} & Q_{k,2r-1} \\ Q_{k,2r-1} & Q_{k,2r-2} \end{bmatrix}$$

From Theorem 1.

$$Q_1^{n+r} = m^{-1}\begin{bmatrix} Q_{k,2n+2r} & Q_{k,2n+2r-1} \\ Q_{k,2n+2r-1} & Q_{k,2n+2r-2} \end{bmatrix}$$

Equating these two matrices,

$$m^{-1}Q_{k,2n+2r} = m^{-2}\left(Q_{k,2n}Q_{k,2r} + Q_{k,2n-1}Q_{k,2r-1}\right)$$
$$mQ_{k,2n+2r} = Q_{k,2n}Q_{k,2r} + Q_{k,2n-1}Q_{k,2r-1}$$

and

$$m^{-1}Q_{k,2n+2r-1} = m^{-2}\left(Q_{k,2n}Q_{k,2r-1} + Q_{k,2n-1}Q_{k,2r-2}\right)$$
$$mQ_{k,2n+2r-1} = Q_{k,2n}Q_{k,2r-1} + Q_{k,2n-1}Q_{k,2r-2}$$

Theorem 4. $mQ_{k,2n-2r} = Q_{k,2n}Q_{k,2r-2} - Q_{k,2n-1}Q_{k,2r-1}$ and

$mQ_{k,2n-2r-1} = Q_{k,2n-1}Q_{k,2r} - Q_{k,2n}Q_{k,2r-1}$, where $n \geq 1, r \geq 1$ and $n \geq r$

Proof.

$$Q_1^{n-r} = Q_1^n (Q_1)^{-r}$$
$$= m^{-1}\begin{bmatrix} Q_{k,2n} & Q_{k,2n-1} \\ Q_{k,2n-1} & Q_{k,2n-2} \end{bmatrix} m \begin{bmatrix} Q_{k,2r} & Q_{k,2r-1} \\ Q_{k,2r-1} & Q_{k,2r-2} \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} Q_{k,2n} & Q_{k,2n-1} \\ Q_{k,2n-1} & Q_{k,2n-2} \end{bmatrix} \frac{1}{Q_{k,2s}Q_{k,2r-2} - Q_{k,2r-1}^2} \begin{bmatrix} Q_{k,2r-2} & -Q_{k,2r-1} \\ -Q_{k,2r-1} & Q_{k,2r} \end{bmatrix}$$
$$= \frac{1}{m^2}\begin{bmatrix} Q_{k,2n}Q_{2r-2} - Q_{k,2n-1}Q_{k,2r-1} & -Q_{k,2n}Q_{2r-1} + Q_{k,2n-1}Q_{k,2r} \\ Q_{k,2n-1}Q_{2r-2} - Q_{k,2n-2}Q_{k,2r-1} & -Q_{k,2n-1}Q_{2r-1} + Q_{k,2n-2}Q_{k,2r} \end{bmatrix}.$$

From Theorem 1. we have

$$Q_1^{n-r} = m^{-1}\begin{bmatrix} Q_{k,2n-2r} & Q_{k,2n-2r-1} \\ Q_{k,2n-2r-1} & Q_{k,2n-2r-2} \end{bmatrix}$$

Therefore, by equating these two matrices, we have

$$m^{-1}Q_{k,2n-2r} = \frac{1}{m^2}Q_{k,2n}Q_{k,2r-2} - Q_{k,2n-1}Q_{k,2r-1},$$
$$mQ_{k,2n-2r} = Q_{k,2n}Q_{k,2r-2} - Q_{k,2n-1}Q_{k,2r-1}$$

and

$$m^{-1}Q_{k,2n-2r-1}=\frac{1}{m^2}Q_{k,2n-1}Q_{k,2r}-Q_{k,2n}Q_{2r-1}$$
$$mQ_{k,2n-2r-1}=Q_{k,2n-1}Q_{k,2r}-Q_{k,2n}Q_{k,2r-1}$$

Theorem 5. For the integers n and r,

$$Q_{k,2n}=m\frac{Q_{k,2n+2r}+Q_{k,2n-2r}}{Q_{k,2r}+Q_{k,2r-2}},n,r\geq 1,n\geq r$$

Proof. From theorem 4

$$mQ_1^{n+r}+mQ_1^{n-r}=Q_1^n(mQ_1^r+mQ_1^{-r})$$

$$=Q_1^n\left[mm^{-1}\begin{pmatrix}Q_{k,2r} & Q_{k,2r-1}\\ Q_{k,2r-1} & Q_{k,2r-2}\end{pmatrix}+m(m^{-1})^{-1}\begin{pmatrix}Q_{k,2r} & Q_{k,2r-1}\\ Q_{k,2r-1} & Q_{k,2r-2}\end{pmatrix}^{-1}\right]$$

$$=mQ_1^{n+r}+mQ_1^{n-r}=Q_1^n(mQ_1^r+mQ_1^{-r})$$ [16]

$$=mQ_1^{n+r}+mQ_1^{n-r}=Q_1^n(mQ_1^r+mQ_1^{-r})$$

But by using Theorem 1.1 and equating two matrices, we have

$$m^{-1}Q_{k,2n}\left(Q_{k,2r}+Q_{k,2r-2}\right)=Q_{k,2n+2r}+Q_{k,2n-2r}$$
$$Q_{k,2n}=m\frac{Q_{k,2n+2r}+Q_{k,2n-2r}}{Q_{k,2r}+Q_{k,2r-2}}$$

Theorem 6. Let some positive integers n, we have

$$\begin{bmatrix}Q_{k,2n+1}\\ Q_{k,2n}\end{bmatrix}=\begin{bmatrix}k^2+1 & k\\ k & 1\end{bmatrix}\begin{bmatrix}Q_{k,2n-1}\\ Q_{k,2n-2}\end{bmatrix}$$

Proof. For $n=1$, the result is insignificant. Let this is true for n. Then we can able to show that it is also true for $n+1$.

$$\begin{bmatrix}Q_{k,2n+3}\\ Q_{k,2n+2}\end{bmatrix}=\begin{bmatrix}k\left(kQ_{k,2n+1}+Q_{k,2n}\right)+Q_{k,2n+1}\\ Q_{k,2n+2}\end{bmatrix}$$
$$=\begin{bmatrix}k^2Q_{k,2n+1}+Q_{k,2n+1}+kQ_{k,2n}\\ kQ_{k,2n+1}+Q_{k,2n}\end{bmatrix}$$
$$=\begin{bmatrix}k^2+1 & k\\ k & 1\end{bmatrix}\begin{bmatrix}k^2+1 & k\\ k & 1\end{bmatrix}\begin{bmatrix}Q_{k,2n-1}\\ Q_{k,2n-2}\end{bmatrix}$$
$$=\begin{bmatrix}k^2+1 & k\\ k & 1\end{bmatrix}\begin{bmatrix}Q_{k,2n+1}\\ Q_{k,2n}\end{bmatrix}$$

Theorem 7.

$$mQ_{k,2n+2r+1} = Q_{k,2r}Q_{k,2n+1} + Q_{k,2r-1}Q_{k,2n}, \quad n \geq 0, r \geq 1$$
$$mQ_{k,2n+2r} = Q_{k,2r-1}Q_{k,2n+1} + Q_{k,2r-2}Q_{k,2n}, \quad n \geq 0, r \geq 1$$

Proof.

$$\begin{aligned} Q_1^{n+r+\frac{1}{2}} &= Q_1^r Q_1^{n+\frac{1}{2}} \\ &= m^{-2}\begin{bmatrix} Q_{k,2r} & Q_{k,2r-1} \\ Q_{k,2r-1} & Q_{k,2r-2} \end{bmatrix}\begin{bmatrix} Q_{k,2n+1} & Q_{k,2n} \\ Q_{k,2n} & Q_{k,2n-1} \end{bmatrix} \\ &= m^{-2}\begin{bmatrix} Q_{k,2r}Q_{k,2n+1} + Q_{k,2r-1}Q_{k,2n} & Q_{k,2r}Q_{k,2n} + Q_{k,2r-1}Q_{k,2n-1} \\ Q_{k,2r-1}Q_{k,2n+1} + Q_{k,2r-2}Q_{k,2n} & Q_{k,2r-1}Q_{k,2n} + Q_{k,2r-2}Q_{k,2r-1} \end{bmatrix} \end{aligned}$$

From Theorem 1.1

$$Q_1^{n+r+\frac{1}{2}} = m^{-1}\begin{bmatrix} Q_{k,2n+2r+1} & Q_{k,2n+2r} \\ Q_{k,2n+2r} & Q_{k,2n+2r-1} \end{bmatrix}$$

Equating these two matrices,

$$m^{-1}Q_{k,2n+2r+1} = m^{-2}\left(Q_{k,2r}Q_{k,2n+1} + Q_{k,2r-1}Q_{k,2n}\right)$$
$$mQ_{k,2n+2r+1} = Q_{k,2r}Q_{k,2n+1} + Q_{k,2r-1}Q_{k,2n}$$

and

$$m^{-1}Q_{k,2n+2r} = m^{-2}\left(Q_{k,2r-1}Q_{k,2n+1} + Q_{k,2r-2}Q_{k,2n}\right)$$
$$mQ_{k,2n+2r} = Q_{k,2r-1}Q_{k,2n+1} + Q_{k,2r-2}Q_{k,2n} \text{ [16]}$$

Properties of the Generalized K-Fibonacci Sequence [19]

Closed-form Expression: The sequence can be expressed using eigenvalues and eigenvectors.

Matrix Trace Relationship: The trace of matrix provides a direct link to the growth factor:

Applications:

Algorithm Optimization: The generalized sequence has applications in optimizing recursive algorithms, particularly in scenarios involving exponential growth.

Cryptography: The unpredictability of matrix-based sequences finds utility in cryptographic key generation.

Modeling Natural Phenomena: Extended Fibonacci sequences model branching patterns and population dynamics with customized growth rates.

Conclusion

The generalized K-Fibonacci sequence of and matrices extends classical Fibonacci theory into matrix algebra, uncovering rich mathematical structures and practical applications. Future work may explore higher-dimensional matrix generalizations and their implications in computational mathematics.

References

[1] T. Koshy, Fibonacei and Lucas Numbers with Applications, John Wiley & Sons, 2011.

[2] G. Cerda-Morales, On generalized Fibonacei and Lucas numbers by matrix methods, Hacet. J. Math. Stat. 42(2) (2013), 173-179.

[3] A. D. Godase and M. B. Dhakne, On the properties of generalized k-Fibonaccilike sequence using matrices, Indian Journal in Number Theory I (2014), 35-42.

[4] D. Kalman, Generalized Fibonacci numbers by matrix methods, Fibonacci Quart. 20(1) (1982), 73-76.

[5] Z. Akyuz and S. Halici, some identities deriving from the nth power of a special matrix, Advances in Difference Equations 2012(1) (2012), 223.

[6] A. Borges, P. Catarino, A. P. Aires, P. Vasco and H. Campos, Two-by-two matrices involving k-Fibonacci and k-Lucas sequences, Appl. Math. Sci. 8(34) (2014), 1659-1666.

[7] Henry W. Gould, A history of the Fibonacci Q-matrix and a higher-dimensional problem, Fibonacci Quart. 19(3) (1981), 250-257.

[8] Douglas A. Lind, The Q matrix as a counterexample in group theory, Fibonacci Quart. 5 (1967), 44.

[9] Charles H. King, Some properties of the Fibonacci numbers, Master's Thesis, San Jose State College, June 1960.

[10] S. L. Basin and V. E. Hoggatt, Jr., A primer for the Fibonacci sequence, Part II, Fibonacci Quart. 1(2) (1963), 61-68.

[11] J. R. Silvester, Fibonacci properties by matrix methods, The Mathematical Gazette 63(425) (1979), 188-191.

[12] R. A. Rosenbaum, An application of matrices to linear recursion relations, Amer. Math. Monthly 66 (1959), 792-793.

[13] E. P. Miles, Jr., Generalized Fibonacci numbers and associated matrices, Amer. Math. Monthly 67 (1960), 745-752.

[14] J. Ivie, A general Q-matrix, Fibonacci Quart. 10(3) (1972), 255-264.

[15] J. L. Brenner, Lucas matrix, Amer. Math. Monthly 58 (1951), 220-221.

[16] Nasir Ahmad and Leena Prasher, Q_1 and Q_2 matrix representations of generalized k-Fibonacci sequence, JP Journal of Algebra, Number Theory and Applications 54 (2022), 19-34. http://dx.doi.org/10.17654/0972555522013

[17] Kumar, N. K., & Sahani, S. K. (2024). Matrices of Fibonacci Numbers. Mikailalsys Journal of Mathematics and Statistics, 3(1), 71-80. https://doi.org/10.58578/mjms.v3i1.4398

[18] Kumar, N. K. (2023). Determination of the Eigen Values of Nurlund Matrix as an operator on Sequence Spaces C and C_0. SSRN: DOI: 10.2139/ssrn.4615935

[19] Kumar, N. K., & Sahani, S. K. (2024). Matrices of Fibonacci Numbers. Mikailalsys Journal of Mathematics and Statistics, 3(1), 71-80. https://doi.org/10.58578/mjms.v3i1.4398