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ON TOPOLOGICAL PROPERTIES OF FUZZY SEQUENCE SPACE THROUGH ORLICZ FUNCTION

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Abstract

In this study, a new class of fuzzy number sequences is defined using Orlicz functions, and various useful classes with a variety of structures have been imposed. The behavior of these new classes has been examined as well as their topological properties and relationship to other fuzzy sequences.

Keywords: Fuzzy Sets, Sequence Spaces, Functional Analysis, Fuzzy Topology, Mathematical Analysis

Introduction

In functional analysis and related mathematical fields, a sequence space is a vector space consisting of infinite sequences of real or complex numbers [9]. Let ω be the set of all functions from the set of natural numbers ℕ to the set of real or complex numbers. Then ω can be turned into a vector space and the subspace of X of ω is called a sequence space. Sequence space is a vector space whose elements are infinite scalars of real or complex numbers and is closed under coordinate-wise addition and scalar multiplication. Numerous scholars have deeply examined the fuzzy set and fuzzy real numbers in a variety of sequence spaces. Zadeh [20] was the first to introduce the idea of fuzzy sets and fuzzy set operations. Since then, a number of authors have covered a variety of topics related to the theory and applications of fuzzy sets, including fuzzy topology. In the field of pure mathematics, a lot of researchers have worked in the field of sequence spaces. Bounded and convergent sequences of fuzzy numbers were first proposed by Motolka [5], who also looked at some of their characteristics. Nanda [8] defined a new metric for demonstrating the completeness of a space of a convergent and bounded sequence of fuzzy real numbers. The double convergent sequence of fuzzy numbers was presented and studied by Mursaleen[7], who also demonstrated that the space containing all double convergent sequences of fuzzy numbers is complete. A few novel sequence spaces of fuzzy numbers were produced by Basarir and Mursaleen [2].

Tripathy [19] and others have researched this further in detail. The idea of convergence in probability is strongly connected to statistical convergence. The material that is currently available on statistical convergence seems to be limited to real or complex analysis. Numerous areas of research in mathematics, management science, and social science have effectively used fuzzy sets and fuzzy logic theories. Altay and Basar[1] defined the double sequence spaces in 2005 and looked at a few of their characteristics and demonstrated that the spaces are complete paranormed or normed spaces. In 2020, Dabbas and Battor examined the characteristics of convergent, null, and limited double sequence spaces defined by the double Orlicz function. The novel double sequence space formed by the Orlicz function is introduced and studied by Paudel et al [11], along with some of its properties such as linear space structure, completeness, and solidity, using the notion of fuzzy real numbers. Similarly in 2023, Paudel [15] defined the difference sequence space $\mathcal{S}(\mathcal{F}, M, \alpha, P)$ of fuzzy real numbers by using the Orlicz function and examined some of its linear topological features, and showed that it is complete by establishing a new paranorm on it. Paudel et al.[13] explored the generalized difference sequence space of fuzzy real numbers in 2022. In 2023, with the help of the Orlicz function, Paudel et al [16] defined the sequence spaces $l_M(X, \bar{\lambda}, \bar{p})$ and $l_M(X , \bar{\lambda}, \bar{p} , L)$ and examined some linear topological characteristics of the spaces using the concepts of fuzzy sets and fuzzy real numbers. In addition, they have established a paranorm to demonstrate the completion of space $l_M(X, \bar{\lambda}, \bar{p}, L)$. Fuzzy logic and fuzzy set theory are widely used in applicable fields of mathematics. Paudel and et al [17] have investigated how Sanchez's medical theory may be applied in medical diagnostics and developed a fuzzy arithmetic-based system for recognizing medical conditions for better treatment. Not only this, Paudel and et al [10] applied fuzzy set theory in making decisions while selecting the best candidate from a group of individuals in the same environment using the maximum-minimum composition.

In this paper a novel class of fuzzy number sequences is developed using Orlicz functions, and some of its topological characteristics have been studied. Basic concept and related definition related to our work are given below.

Definition and Preliminaries:

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing, and convex with $M(0) = 0, M(x) > 0$ for $x > 0$, and $M(x) \to 0$ as $x \to \infty$. If the convexity of Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called the modulo function.

Wladyslaw Orliczfirstly introduced Orlicz space in 1932. Later on, Lindenstrauss and Tzafriri [9] used the idea of the Orlicz function to construct the sequence space

$$l_M = \left\{ x \in \omega: \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \; for\ some\ \rho > 0 \right\}$$

The space l_M with the norm $\|x\|$ is defined by

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

becomes a Banach space and is called Orlicz sequence space.

An Orlicz function *M* is said to satisfy Δ_2 −condition for all values of *u* if there exists *k* >0 such that $M(2u) \leq kM(u), u \geq 0.$

Let D be the set of all bounded intervals A= [a, b] on the real line ℝ. Then for any $A, B \in D$ with A = $[a_1,b_1]$ and $B = [a_2, b_2]$ then A ≤ B if $a_2 \leq a_1$ and $b_1 \leq b_2$.

Define a relation *d* on D by $d(A, B) = max\{|a_2 - a_1|, |b_2 - b_1|\}$

Then clearly, *d* defines a metric in D, and obviously (D, d) is a complete metric space.

Let U be the universe of discourse and $X \subseteq U$. A fuzzy set X in *U* is defined as the collection of order pair (*x*, $\mu_X(x)$) where μ_X : U →[0, 1] and $x \in$ U. Here $\mu_X(x)$ is called the degree of membership of *x*.

A fuzzy number is a very set on the real axis, i.e. a mapping $X : \mathbb{R} \rightarrow [0, 1]$which satisfies the following four conditions

i. X is normal i. e. there exists $x^o \in \mathbb{R}$ such that $X(x^o) = 1$;
ii. X is fuzzy convex i. e. for $x, y \in \mathbb{R}$ and $0 \leq \alpha \leq 1, X(\alpha x + (1-\alpha)y) \geq \min[X(x), X(y)]$
iii. X is upper semi-continuous ;
iv. The closure of the set $C(\mathbb{R}) = \{x \in \mathbb{R}: X(x) \geq 0\}$ is compact, and the set has a linear structure introduced by operators

$A + B = \{a + b: a \in A, b \in B\}$ and $\gamma A = \{\gamma\, a : a \in A\}$

For $A, B \in C(\mathbb{R})$ and $\gamma \in \mathbb{R}$,the Hausdroff distance between A and B of $C(\mathbb{R})$ is defined as

$$\delta_\infty(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|\ , \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}$$

where,$\|.\|$ denotes the usual Euclidean norm on ℝ. It is well known that $(C(\mathbb{R}), \delta_\infty)$ is complete metric space.

Let $\lambda \in (\lambda_{mn})$ be a non-decreasing sequence of positive real numbers tending to infinite such that

$$\lambda_{m+1,n} \leq \lambda_{m,n+1}, \lambda_{m,n+1} \leq \lambda_{m+1,n}$$

$$\lambda_{m,n} - \lambda_{m+1,n} \leq \lambda_{m,n+1} - \lambda_{m+1,n+1}, m_{1,1} = 1$$

And

$$\mathrm{I}_{m,n} = \{(k,l) : m - \lambda_{m,n+1} \le k \le m, \qquad n - \lambda_{m,n+1} \le l \le n\}$$

The generalized double Delvalle-Poussin mean is defined by

$$\mathrm{t}_{m,n} = t_{m,n}(\lambda_k, 1) = \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in \mathrm{I}_{m,n}} (X_{k,l})$$

A double sequence of fuzzy numbers $X = (X_{k,l})$ is a function form $X: \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathrm{I})$, the set of fuzzy real numbers. The fuzzy number $X_{k,l}$ is the value of the function at the point $(k, l) \in \mathbb{N} \times \mathbb{N}$ and is called (k, l)- term of the double sequence. Let $S''(F)$ denote the set of all double sequences of fuzzy numbers.

A double sequence $X = (X_{k,l})$ of fuzzy number is said to be convergent in the Pring Sheims sense or P- convergent to a fuzzy number X_o if for every $\varepsilon > 0$, there exists $\mathbb{N} \in Z_+$ such that $d(X_{k,l}, X_o) < \varepsilon$ for all $k, l \ge \mathbb{N}$, and we write P-$\lim X = X_o$. The number X_o is called Pring-Sheims limit of $(X_{k,l})$. More exactly, we say that a double sequence $(X_{k,l})$ converges to a finite number X_o if $X_{k,l}$ tends to X_o as both k and l to ∞ independently of one other.

A double sequence $X = (X_{k,l})$ of fuzzy number is said to be λ −statistically convergent to X_o provided that for every $\varepsilon > 0$,

$$\mathrm{P} - \lim_{m,n} \frac{1}{m,n} \left|\{(j,k): j \le m \text{ and } k \le n : d(X_{k,l}, X_o) \ge \varepsilon\}\right| = 0$$

We denote the set of all double λ −statistically convergent sequence of fuzzy numbers is denoted by $S''(\lambda)^F$.

Let $M = (M_{k,l})$ be a sequence of Orlicz functions, $P = (p_k)$ be a bounded sequence of positive real numbers, and $\mathrm{u} = (\mathrm{u}_k)$ be a sequence of strictly positive real numbers. We define the following classes for sequences

$$\omega''(\lambda, M, u, P)^F = \left\{ X = (X_{k,l}) \in \right.$$

$$\left. S''(F): \lim_{m,n\to\infty} \frac{1}{\lambda_{m,n}} \sum_{k,l\in I_{m,n}} u_{k,l} \left[M_{k,l} \left(\frac{(d(t_{mn}(X_{k,l}), X_o)}{\rho} \right) \right]^{p_{k,l}} = 0, \quad uniformaly\ in\ m,n\ for\ some\ \rho > 0 \right\}$$

$$\omega''_o(\lambda, M, u, P)^F = \left\{ X = (X_{k,l}) \in S''(F): \lim_{m,n\to\infty} \frac{1}{\lambda_{m,n}} \sum_{k,l\in I_{m,n}} u_{k,l} \left[M_{k,l} \left(\frac{\left(d\left(t_{mn}(X_{k,l}), \bar{0}\right)\right)}{\rho} \right) \right]^{p_{k,l}} = 0, \ uniformaly\ in\ m,n\ for\ some\ \rho > 0 \right\}$$

$$\omega''_\infty(\lambda, M, u, P)^F = \left\{ X = (X_{k,l}) \in S''(F): \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l\in I_{m,n}} u_{k,l} \left[M_{k,l} \left(\frac{\left(d\left(t_{mn}(X_{k,l}), \bar{0}\right)\right)}{\rho} \right) \right]^{p_{k,l}} < \infty \ uniformalyin\ m,n\ for\ some\ \rho > 0 \right\}$$

where, $\bar{0}(t) = \begin{cases} 1 & t = (0,0,0,\dots\dots\dots\dots,0) \\ 0 & otherwise \end{cases}$

If X $\in \omega''(\lambda, M, u, P)^F$, we say that X is strongly almost λ −convergent with respect to the Orlilcz function. In this case we write $X_{k,l} \to X$.

The following classes are defined by giving particular values of *M, u, P,* and λ

(i) For $\lambda_{mn} = 1$, we write

$\omega''(\lambda, M, u, P)^F = \omega''(M, u, P)^F$, $\omega''_o(\lambda, M, u, P)^F = \omega''_o(M, u, P)^F$, and

$\omega''_\infty(\lambda, M, u, P)^F = \omega''_\infty(M, u, P)^F$

(ii) If $M = (M_{k,l})(x) = x$ for all values of *k, l* we write

$\omega''(\lambda, M, u, P)^F = \omega''(\lambda, u, P)^F$, $\omega''_o(\lambda, M, u, P)^F = \omega''_o(\lambda, u, P)^F$ and

$\omega''_\infty(\lambda, M, u, P)^F = \omega''_\infty(\lambda, u, P)^F$

(iii) If $p_{k,l} = 1$ for all $k, l \in \mathbb{N}$,then

$\omega''(\lambda, M, u, P)^F = \omega''(\lambda, M, u)^F$, $\omega''_o(\lambda, M, u, P)^F = \omega''_o(\lambda, M, u)^F$ and

$\omega''_\infty(\lambda, M, u, P)^F = \omega''_\infty(\lambda, M, u)^F$

(iv) If $M = M_{k,l}(x) = x$, $p_{k,l} = 1$ for all $k, l \in \mathbb{N}$,then we write

$\omega''(\lambda, M, u, P)^F = \omega''(\lambda, u)^F$, $\omega''_o(\lambda, M, u, P)^F = \omega''_o(\lambda, u)^F$ and

$$\omega''_\infty(\lambda, M, u, P)^F = \omega''_\infty(\lambda, u)^F$$

(v) If $p_{k,l} = 1$, $u_{k,l} = 1$ for all $k, l \in \mathbb{N}$ then we write $\omega''(\lambda, M, u, P)^F = \omega''(\lambda, M)^F$, $\omega''_o(\lambda, M, u, P)^F = \omega''_o(\lambda, M)^F$, and $\omega''_\infty(\lambda, M, u, P)^F = \omega''_\infty(\lambda, M)^F$

(vi) If $M = M_{k,l}(x) = x$, $p_{k,l} = 1$, $u_{k,l} = 1$ for all $k, l \in \mathbb{N}$ then we wri $\omega''(\lambda, M, u, P)^F = \omega''(\lambda)^F$, $\omega''_o(\lambda, M, u, P)^F = \omega''_o(\lambda)^F$, and $\omega''_\infty(\lambda, M, u, P)^F = \omega''_\infty(\lambda)^F$

In this paper following inequality will be used

Let $P = (p_{k,l})$ be a double sequence of positive real numbers with $0 < p_{k,l} \leq \sup_{k,l} p_{k,l} = H$ and let $D = \max\{1, 2^{H-1}\}$. Then, for the factorial sequences a_k and b_k of scalars, we have

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \leq D\left(|a_{k,l}|^{p_{k,l}} + |a_{k,l}|^{p_{k,l}}\right)$$

Main Result

Theorem 1: Suppose $M = (M_{k,l})$ be a sequence of Orlicz function , $P = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be a sequence of positive numbers, then $\omega''_o(\lambda, M, u, P)^F \subset \omega''(\lambda, M, u, P)^F \subset \omega''_\infty(\lambda, M, u, P)^F$.

Proof: The relation $\omega''_o(\lambda, M, u, P)^F \subset \omega''(\lambda, M, u, P)^F$ is obvious. Let $X \in \omega''(\lambda, M, u, P)^F$ then we get,

$$\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \left[M_{k,l}\left(\frac{(d(t_{mn}(X), \bar{0})}{2\rho}\right)\right]^{p_{k,l}} \leq$$

$$\frac{k}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \frac{1}{2^{p_{k,l}}} u_{k,l} \left[M_{k,l}\left(\frac{(d(t_{mn}(X), X_o)}{\rho}\right)\right]^{p_{k,l}} +$$

$$\frac{k}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \frac{1}{2^{p_{k,l}}} u_{k,l} \left[M_{k,l}\left(\frac{(d(X, \bar{0})}{\rho}\right)\right]^{p_{k,l}}$$

$$\leq \frac{k}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \frac{1}{2^{p_{k,l}}} u_{k,l} \left[M_{k,l}\left(\frac{(d(t_{mn}(X), X_o)}{\rho}\right)\right]^{p_{k,l}} +$$

$$\mathrm{k} \max_{k,l \in I_{mn}} \left\{\max\left\{1, \sup u_{k,l} \left[M_{k,l}\left(\frac{d(X_o, \bar{0})}{\rho}\right)\right]^{H_1}\right\}\right\}$$

where, $\sup_{k,l} p_{k,l} = H_1$ and $k = \max(1,\ 2^{H_1-1})$. Thus $X \in \omega''_\infty(\lambda, M, u, P)^F$.

Theorem 2: Suppose $M = (M_{k,l})$ be a sequence of Orlicz function $P = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of positive real numbers then, the classes $\omega''_o(\lambda, M, u, P)^F$, $\omega''(\lambda, M, u, P)^F$, $\omega''_\infty(\lambda, M, u, P)^F$ are linear.

Proof : Let $X = (X_{k,l})$ and $Y = (Y_{k,l})$ are the elements of $\omega''_\infty(\lambda, M, u, P)^F$ and α, β are scalars. Then there exists $\rho_1 > 0,\ \ \rho_2 > 0$ such that

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \left[M_{k,l} \left(\frac{(d(t_{mn}(X), \bar{0})}{\rho_1} \right) \right]^{p_{k,l}} < \infty \text{ uniformly in } m, n$$

and

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \left[M_{k,l} \left(\frac{(d(t_{mn}(Y), \bar{0})}{\rho_2} \right) \right]^{p_{k,l}} < \infty \text{ uniformly in } m, n$$

Define $\rho_3 = \max(2|\alpha|\rho_1,\ 2|\beta|\rho_2)$. Since, $M = (M_{k,l})$ is non-decreasing and convex, we have

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \left[M_{k,l} \left(\frac{(\alpha\, d(t_{mn}(X), \bar{0}) + (\beta\, d(t_{mn}(Y), \bar{0})}{\rho_3} \right) \right]^{p_{k,l}}$$

$$\leq \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \left[M_{k,l} \left(\frac{(\alpha\, d(t_{mn}(X), \bar{0})}{\rho_3} \right) \right.$$

$$\left. + M_{k,l} \left(\frac{(\beta\, d(t_{mn}(Y), \bar{0})}{\rho_3} \right) \right]^{p_{k,l}}$$

$$\leq \frac{1}{2} \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \left[M_{k,l} \left(\frac{(d(t_{mn}(X), \bar{0})}{\rho_1} \right) \right]^{p_{k,l}}$$

$$+ \frac{1}{2} \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \left[M_{k,l} \left(\frac{(d(t_{mn}(Y), \bar{0})}{\rho_2} \right) \right]^{p_{k,l}} < \infty$$

$\Rightarrow \alpha X + \beta Y \in \omega''_\infty(\lambda, M, u, P)^F$. This proves that the class $\omega''_\infty(\lambda, M, u, P)^F$ is a linear space. Similarly we can prove for other class as linear space.

Theorem 3: If $0 \leq p_{k,l} \leq r_{k,l}$ for all $k, l \in \mathbb{N}$ and $\left(\frac{p_{k,l}}{r_{k,l}}\right)$ bounded the $\omega''_{\infty}(\lambda, M, u, r)^F \subseteq \omega''_{\infty}(\lambda, M, u, P)^F$.

Proof: $X = (X_{k,l}) \in \omega''_{\infty}(\lambda, M, u, r)^F$ then,

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \left[M_{k,l}\left(\frac{(d(t_{mn}(X), \bar{0})}{\rho}\right)\right]^{p_{k,l}} < \infty \text{ uniformly in } m, n$$

Let $s_{k,l} = \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \left[M_{k,l}\left(\frac{(d(t_{mn}(X), \bar{0})}{\rho}\right)\right]^{p_{k,l}}$ and $\lambda_{k,l} = \frac{p_{k,l}}{r_{k,l}}$

Since $p_{k,l} \leq r_{k,l}$ we have $0 \leq \lambda_{k,l} \leq 1$. Take $0 < \lambda < \lambda_{k,l}$

Define, $u_{k,l} = \begin{cases} s_{k,l} & if \ s_{k,l} \geq 1 \\ 0 & if \ s_{k,l} < 1 \end{cases}$ and $v_{k,l} = \begin{cases} 0 & if \ s_{k,l} \geq 1 \\ s_{k,l} & if \ s_{k,l} < 1 \end{cases}$

Then $s_{k,l} = u_{k,l} + v_{k,l}$, $s_{k,l}^{\lambda_{k,l}} = u_{k,l}^{\lambda_{k,l}} + v_{k,l}^{\lambda_{k,l}}$

It follows that,

$u_{k,l}^{\lambda_{k,l}} \leq u_{k,l} \leq s_{k,l}$, $v_{k,l}^{\lambda_{k,l}} \leq v_{k,l}^{\lambda}$

Since, $s_{k,l}^{\lambda_{k,l}} = u_{k,l}^{\lambda_{k,l}} + v_{k,l}^{\lambda_{k,l}}$ then $s_{k,l}^{\lambda_{k,l}} = s_{k,l} + v_{k,l}^{\lambda}$

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \left[M_{k,l}\left(\frac{(d(t_{mn}(X), \bar{0})}{\rho}\right)^{r_{k,l}}\right]^{\lambda_{k,l}}$$

$$\leq \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \left[M_{k,l}\left(\frac{(d(t_{mn}(X), \bar{0})}{\rho}\right)\right]^{r_{k,l}}$$

$$\Rightarrow \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \left[M_{k,l}\left(\frac{(d(t_{mn}(X), \bar{0})}{\rho}\right)^{r_{k,l}}\right]^{\frac{p_{k,l}}{r_{k,l}}}$$

$$\leq \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \left[M_{k,l}\left(\frac{(d(t_{mn}(X), \bar{0})}{\rho}\right)\right]^{r_{k,l}}$$

$$\Rightarrow \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \left[M_{k,l}\left(\frac{(d(t_{mn}(X), \bar{0})}{\rho}\right)\right]^{p_{k,l}}$$

$$\leq \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \left[M_{k,l} \left(\frac{(d(t_{mn}(X), \bar{0})}{\rho} \right) \right]^{r_{k,l}}$$

But,

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \left[M_{k,l} \left(\frac{(d(t_{mn}(X), \bar{0})}{\rho} \right) \right]^{r_{k,l}} < \infty \text{ uniformaly in } m,n$$

Therefor,

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \left[M_{k,l} \left(\frac{(d(t_{mn}(X), \bar{0})}{\rho} \right) \right]^{p_{k,l}} < \infty \text{ uniformaly in } m,n$$

And hence $X \in \omega''_{\infty}(\lambda, M, u, P)^F$. Thus, $\omega''_{\infty}(\lambda, M, u, r)^F \subseteq \omega''_{\infty}(\lambda, M, u, P)^F$.

Proved

Theorem4: Suppose $M = (M_{k,l})$ be a sequence of Orlicz function, $\mathrm{P} = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$be a sequence of strictly positive real numbers. If $\sup_{k,l} (M_{k,l}(x))^{p_{k,l}} < \infty$ for all fixed x >0 $\omega''(\lambda, M, u, P)^F \subset \omega''_{\infty}(\lambda, M, u, P)^F$

Theorem 5: Let $M = (M_k)$ be a sequence of Orlicz functions, $X = (X_{k,l})$ be a double bounded sequence of fuzzy numbers and $0 < h = \inf p_{k,l} \leq \sup p_{k,l} = H_1 < \infty$. Then $s''(\lambda)^F \subset \omega''(\lambda, M, u, P)^F$.

Proof: Suppose that $X \in l_{\infty}^F$ and $X_{k,l} \longrightarrow X_o (s''(\lambda))^F$. Since $X \in l_{\infty}^F$, there exists a constant $K > 0$ such that $d(t_{mn}(X), X_o) < K$ for all $m, n \in \mathbb{N}$.

Let, $\varepsilon > 0$ be given. We have

$$\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l} \left[M_{k,l} \left(\frac{(d(t_{mn}(X), X_o)}{\rho} \right) \right]^{p_{k,l}}$$

$$\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n,d(t_{mn}(X), X_o) \geq \varepsilon}} u_{k,l} \left[M_{k,l} \left(\frac{(d(t_{mn}(X), \bar{0}))}{\rho} \right) \right]^{p_{k,l}} +$$

$$\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n,d(d(t_{mn}(X),X_o)<\varepsilon}} u_{k,l}\left[M_{k,l}\left(\frac{(d(t_{mn}(X),\bar{0}))}{\rho}\right)\right]^{p_{k,l}}$$

$$\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n,d(t_{mn}(X),X_o)\geq\varepsilon}} max\left\{u_{k,l}\left[M_{k,l}\left(\frac{K'}{\rho}\right)\right]^{h}, u_{k,l}\left[M_{k,l}\left(\frac{K'}{\rho}\right)^{H_1}\right]\right\} +$$

$$\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n,(d(t_{mn}(X),X_o)<\varepsilon}} u_{k,l}\left[M_{k,l}\left(\frac{\varepsilon}{\rho}\right)\right]^{p_{k,l}}$$

$$\leq \max_{k,l \in I_{m,n}}\left\{u_{k,l}[M_{k,l}(T)]^{h}, u_{k,l}[M_{k,l}(T)^{H_1}]\right\} \frac{1}{\lambda_{mn}}\left|\{(k,l) \in I_{m,n} : d(t_{mn}(X),X_o) \geq \varepsilon\}\right|$$

$$+ \max_{k,l \in I_{m,n}}\left\{u_{k,l}[M_{k,l}(\varepsilon_1)]^{h}, u_{k,l}[M_{k,l}(\varepsilon_1)]^{H_1}\right\}$$

where, $T = \frac{K\prime}{\rho}$, $\frac{\varepsilon}{\rho} = \varepsilon_1$ and hence $X \in \omega''(\lambda, M, u, P)^F$. This completes the proof.

Theorem 6: Let $M = (M_k)$ be a sequence of Orlicz functions, $X = (X_{k,l})$ be a double bounded sequence of fuzzy numbers and $0 < h = \inf p_{k,l} \leq p_{k,l} \leq \sup p_{k,l} = H_1 < \infty$. Then $''(\lambda, M, u, P)^F \subset s''(\lambda)^F$.

Proof : The proof of the theorem follows from the following inequality

$$\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} u_{k,l}\left[M_{k,l}\left(\frac{(d(t_{mn}(X),X_o)}{\rho}\right)\right]^{p_{k,l}}$$

$$= \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n,d(t_{mn}(X),X_o)\geq\varepsilon}} u_{k,l}\left[M_{k,l}\left(\frac{d(t_{mn}(X),X_o)}{\rho}\right)\right]^{p_{k,l}} +$$

$$\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n,d(t_{mn}(X),X_o)<\varepsilon}} u_{k,l}\left[M_{k,l}\left(\frac{d(t_{mn}(X),X_o)}{\rho}\right)\right]^{p_{k,l}}$$

$$\geq \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n,d(t_{mn}(X),X_o)\geq\varepsilon}} u_{k,l}\left[M_{k,l}\left(\frac{d(t_{mn}(X),X_o)}{\rho}\right)\right]^{p_{k,l}}$$

$$\geq \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n,d(t_{mn}(X),X_O) \geq \varepsilon}} \min \left\{ u_{k,l}\left[M_{k,l}(\varepsilon_1)\right]^{h_1}, \quad u_{k,l}\left[M_{k,l}(\varepsilon_1)^{H_1}\right] \right\}$$

$$\geq \frac{1}{\lambda_{mn}} \left| \left\{ (k,l) \in I_{m,n} : d(t_{mn}(X), X_o) \geq \varepsilon \right\} \right|$$

$$+ \min_{k,l \in I_{m,n}} \left\{ u_{k,l}\left[M_{k,l}(\varepsilon_1)\right]^{h}, u_{k,l}\left[M_{k,l}(\varepsilon_1)\right]^{H_1} \right\}$$

where, $\frac{\varepsilon}{\rho} = \varepsilon_1$.

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