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CONCEPT OF BILINEAR TRANSFORMATION, JACOBIAN AND CONFORMAL MAPPING WITH APPLICATIONS

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Abstract

This article explores the concepts of bilinear transformation, Jacobian transformation, and conformal mapping, focusing on their essential properties and presenting key results. The discussion revolves around isogonal transformation, conformal transformation, Jacobian transformation, and bilinear transformation, as well as critical points and fixed points.

Keywords: Isogonal transformation, Conformal transformation, Jacobian transformation, Bilinear transformation, Critical point, Fixed point

Introduction

If f(x) is a real-valued function of the real variable x,

$$f : x \rightarrow y \tag{1}$$

A mapping or transformation of points in the z-plane is the relationship specified by equation (1) between two points in the z-plane and w plane. Images of each other refer to the correspondence set of points in the two planes. The equations

$$u = u(x, y)\ ,\ v = v(x, y) \tag{2}$$

are referred to as transformations [2]. This article deals with conformal mapping, Jacobians, bilinear transformations, fixed points, and normal [1,2,3].

Bilinear Transformation

It is also called linear fractional transformation. The transformation T defined as

$$w = T(z) = \frac{az+b}{cz+d} \tag{3}$$

a,b,c, and d are complex constants , ad - bc$\neq 0$ is known as bilinear transformation. The constant ad-bc is determinant of bilinear transformation. Now , if ad-bc= 1 then it is normalized. This transformation of (i) can be written as

$$cwz+dw-az-b=0 \tag{4}$$

The equation (ii) is linear both in z and w, so it is bilinear transformation. It is also called Mobius transformation, who first studied the same. Complex numbers a, b, c, and d are called co-efficient of Mobius Transformation of S(z). The determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is called the determinant of Mobius Transformation S(z). The constants a, b, c, d do not uniquely determine since, $\lambda \neq 0$ for any,

$$S(z) = \frac{(\lambda a)z+(\lambda b)}{(\lambda c)z+(\lambda d)}$$

$f' = \frac{\partial f}{\partial z} = \frac{dw-b}{-cw+a}$ does not vanish, the Möbius transformation f(z) is conformal at every point except its pole $z = -\frac{d}{c}$. The inverse function $z = f^{-1}(w)$, ($f \circ f^{-1} \equiv I$, where I is the identity, can be computed as: $f^{-1}(w) = \frac{dw-b}{-cw+a}$ [4].

Product of two bilinear transformations

Suppose two transformations T_1 and T_2 defined are

$$T_1(z) = \frac{a_1 z+b_1}{c_1 z+d_1},\quad (a_1 d_1 - b_1 c_1 \neq 0) \tag{5}$$

$$T_1(\zeta) = \frac{a_2\zeta\ + b_2}{c_2\ \zeta\ + d_2}, \quad (a_2 d_2 - b_2 c_2 \neq 0) \tag{6}$$

In equation (5), there is one- one correspondence between the z-plane and ζ -plane. The transformation (ii) also maps a one to one correspondence between the z-plane and ζ -plane. Now defining a transformation from z-plane to w-plane from the relation

$$\text{w} = T_2(T_1(z)) \tag{7}$$

$$T_2(T_1(z)) = T_2\left\{\frac{a_1 z + b_1}{c_1 z + d_1}\right\} \quad \text{from equation (5)}$$

$$= \frac{a_2\left(\frac{a_1 z+b_1}{c_1 z+d_1}\right)+b_2}{c_2\left(\frac{a_1 z+b_1}{c_1 z+d_1}\right)+d_2} = \frac{(a_2 a_1 + b_2 c_1)\ z + (a_2 b_1 + b_2 d_1)}{(c_2 a_1 + d_2 c_1)\ z + (c_2 b_1 + d_2 d_1)} \quad [\text{ From 6}]$$

$$\text{w} = T_2(T_1(z)) \frac{\alpha z+\beta}{\gamma z+\delta}$$

(8)

This equation (8) denotes a bilinear transformation and is called resultant or the product of the two transformations.

Bilinear transformation with simple geometric properties [2]

Suppose the bilinear transformation

w = $\frac{az+b}{cz+d}$, where ad- bc $\neq 0$, and c $\neq 0$. This can be

It may be written as

$$\text{W} = \frac{a\left(z+\frac{d}{c}\right)+b-\frac{ad}{c}}{c(z\frac{d}{c})} = \frac{a}{c} + \frac{bc-ad}{c} \cdot \frac{1}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c^2} \cdot \frac{1}{z+\frac{d}{c}}$$

This transformation can be considered as the combination of following three transformations

$z_1 = z + \frac{d}{c}, z_2 = \frac{1}{z_1}$ *and* $z_3 = \frac{bc-ad}{c^2} z_2$ so that

w = $\frac{a}{c} + z_3$.This transformation of the form z_1.Three auxiliary transformation are of the form W= z + α, w = $\frac{1}{2}$, w = βz. So, the bilinear transformation is the resultant of bilinear transformation of the form w= z + α, w = $\frac{1}{2}$, w = βz

Theorem1. The set of all bilinear transformation builds a non-abelian group under the product of transformations.

Proof. The set of bilinear transformations satisfy the properties of group.

Associativity. $(T_1T_2)T_3 = T_1(T_2T_3)$

Existence of identity. The identity mapping, I defined by w=I(z) = z is a bilinear transformation so that I serves as an identity element.

Existence of inverse. The inverse of the transformation

W = T(z) = $\frac{az+b}{cz+d}$ is $z = T^{-1}(w) = \frac{dw-b}{-cw+a}$

$$T^{-1}T(z) = T^{-1}\left(\frac{az+b}{cz+d}\right) = \frac{d\left(\frac{az+b}{cz+d}\right)-b}{-c\left(\frac{az+b}{cz+d}\right)+a} = \frac{adz+db-bcz-bd}{-caz-cb+caz+ad} = \frac{(ad-bc)z}{ad-bc} = z$$

= $TT^{-1}(w)$ = w. The set of all bilinear transformations builds a group under

the product of transformations. It is remarkable that $T_1T_2(Z) \neq T_2T_1(Z)$

Fixed points of bilinear transformation [2]

The points that coincide with their transformations under a bilinear transformation are said to be fixed points.

Let the bilinear transformation defined by

w = $\frac{az+b}{cz+d}$ (ad- bc $\neq 0$)

(9)

Now substituting w=z to get the fixed points of this transformation

z = $\frac{az+b}{cz+d}$

or, c$z^2 + (d-a)z - b$ = 0

(10)

Case I. Let c $\neq 0$, the roots of equation (ii) are

z = $\frac{(a-d)\pm\sqrt{[(d-a)^2+4bc\,]}}{2c}$ gives one or two finite fixed points according to

$(d-a)^2 + 4bc$ = 0 or $\neq 0$

Case II. Let c = 0 that d $\neq 0$

The transformation becomes w = $\frac{a}{d}z + \frac{b}{d}$. The other fixed point is

z = $\frac{a}{d}z + \frac{b}{d}$ or, (a-d) z+b = 0 . Hence if a – d ≠0, then two fixed points ∞

and$\frac{b}{d-a}$ but a-d =0 gives only fixed point ∞.

Normal form of bilinear transformation [2]

Theorem2. Every bilinear transformation with α, β as fixed points α, β can be put in the form

$$\frac{w-\alpha}{w-\beta} = \lambda \frac{z-\alpha}{z-\beta}.$$

Proof. Let any bilinear transformation with α, β as fixed point and suppose it transforms a point γ into the point δ., then the points α, β, γ, z are mapped into the points α, β, δ, w respectively. Since cross ratio is conserved under a bilinear transformation, we have

(w,$\alpha, \delta, \beta) = (z, \alpha, \gamma, \beta)$ Or, $\frac{(w-\alpha)(\delta-\beta)}{(\alpha-\delta)(\beta-w)} = \frac{(z-\alpha)(\gamma-\beta)}{(\alpha-\gamma)(\beta-z)}$, Conformal Mappings

Or $\frac{w-\alpha}{w-\beta} = \frac{(\alpha-\delta)(\beta-\gamma)}{(\beta-\delta)(\alpha-\gamma)} . \frac{z-\alpha}{z-\beta}$ which is the from

$\frac{w-\alpha}{w-\beta} = \lambda \ \frac{z-\alpha}{z-\beta}$ which $\lambda = \frac{(\alpha-\delta)(\beta-\gamma)}{(\beta-\delta)(\alpha-\gamma)}$.

Theorem 2. Every bilinear transformation has only one fixed point α can be put in the form

$\frac{1}{w-z} = \frac{1}{z-\alpha} + \lambda.$

Proof. Consider the transformation w = $\frac{az+b}{cz+d}$ and let α is the only one finite fixed point.

Then the equation w = $\frac{az+b}{cz+d}$ or, $cz^2 + (d-a)z - b = 0$ ≡ c $(z-\alpha)^2$ has only one root α, so that

Now, d – a = - $2\alpha c$ and –b = cα^2

i.e. a = d+ $2\alpha c$ and b = - cα^2 , we can write w = $\frac{(d-2\alpha c)\, z - c\alpha^2}{cz+d}$

cwz+dw= dz-$2\alpha cz - c\alpha^2$

c(w-α) (z- α)+ $c\alpha w + c\alpha z - c\alpha^2$+ dw = dz+2$\alpha cz - c\alpha^2$, on solving we get

$\lambda = \frac{c}{d+c\alpha} = \frac{c}{d+c\left(\frac{a-d}{2c}\right)} = \frac{2c}{a+d}$.

Different transformations [5]

There are different types of transformations i.e. elliptic, hyperbolic and parabolic transformations.

(i) Elliptic transformations

The transformation

$$w= T(z) = \frac{az+b}{cz+d} \tag{11}$$

has to finite and distinct fixed points α, β if c $\neq 0$ and

$$\Delta= (a-d)^2 + 4bc \neq 0 \tag{12}$$

In this case the transformation T can be put in the from

$$w = \frac{w-\alpha}{w-\beta} = k \frac{z-\alpha}{z-\beta} \tag{13}$$

Where k = $\frac{c\beta+b}{c\alpha+d} \neq 0$ *is* a finite constant.

If $|k| = 1$, the transformation is said to be elliptic. In this case

$$\left|\frac{w-\alpha}{w-\beta}\right| = \left|\frac{z-\alpha}{z-\beta}\right|$$

Every circle of the first kind in the Steiner system with limiting points α, β is left invariant. Under this mapping, then, the point's z "flow" along these invariant circles in such a way that circles of the second kind pass over into one another.

If c = 0. and $\Delta \neq$ 0, the second fixed point is at infinity and the transformation takes the form

$$w - \alpha = k (z- \alpha) \tag{14}$$

where k = $\frac{a}{b}$ = 1. That is, $|a| = |d|$. Here the transformation (ii) represents a rotation (through the angle arg k) about the fixed point α.

(ii) Hyperbolic transformations

Let $\Delta \neq 0,\ c\ \neq 0$ k be real and positive so that arg k =0.

Then (13) gives arg $\left(\frac{w-\alpha}{w-\beta}\right) = \arg k + \arg\frac{z-\alpha}{z-\beta} = arg\,\frac{z-\alpha}{z-\beta}$.

Thus, in this case the circles of the second kind in the Steiner system belonging to the limit to the limit points α, β are left invariant while the circles of the first kind go over into one another. The transformation therefore represents "flow" along the circles of the second kind. A transformation of this type is called hyperbolic. If c = 0, $\Delta \neq 0$, one of the points say $\beta\ is$ of at infinity and the transformation is of the form (3) it is hyperbolic when arg k = arg$\frac{a}{d} =$ $arg\ d = 0$.

(iii) Parabolic transformations.

If $\Delta = (a-b)^2$ + 4bc = D, then he transformation (11) is called Parabolic.

First, let c = 0. Then a = d $\neq 0$ and the linear transformation

(i) get the form

$$w = z + \frac{b}{d},\ \ \frac{b}{d} \neq \infty \tag{15}$$

If $\frac{b}{d}$ = 0, that is, b = 0 then (v) reduces to an identity transformation. Then every point is a fixed point. In case$\frac{b}{d} \neq 0$, Represents a simple translation. In this case the only fixed point is at ∞. The straight linesL_1 in the direction $\frac{b}{d}$ are stream lines, their perpendiculars the lines L_2pass over into one another under the flow. Now let $\Delta = 0, c \neq 0$.

$\alpha = \frac{a-d}{2c}$. This has the only fixed point of the parabolic transformation, since the point z = ∞ is transformed into w = $\frac{a}{c}$ so that ∞ is not a fixed point. In this case the transformation can be represented in the form

$$\frac{1}{w-\alpha} = \frac{1}{z-\alpha} + \lambda \qquad \text{where} \qquad \lambda = \frac{2c}{a+d}$$

Jacobians [1,4,5]

The transformation

w = f(z), i.e. u = u(x, y), v = v(x, y) maps a closed region D of the z- plane into a closed region D' of the w-plane. Suppose Δ_z and Δ_w denote the area of these regions. Also let u and v are continuously differentiable, then $\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z} = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|$ then the determinant

$$= \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}.\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}.\frac{\partial v}{\partial x}$$ is said to be jacobians of the transformation.

If f(z) is an analytic function, then using Cauchy- Riemann equations, then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial x}.\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\left(-\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 = \left|\frac{\partial u}{\partial x} + i\,\frac{\partial u}{\partial y}\right|^2 = |f'(z)|^2 .$$

If $u_1, u_2, u_3 \cdots\cdots u_n$ are functions of n variables of $x_1, x_2, x_3 \cdots\cdots x_n$ then the determinant

$$\begin{vmatrix} \frac{\partial u1}{\partial x1} & \frac{\partial u1}{\partial x2} \cdots\cdots & \frac{\partial u1}{\partial xn} \\ \frac{\partial u2}{\partial x1} & \frac{\partial u2}{\partial x2} \cdots\cdots & \frac{\partial u2}{\partial xn} \\ \frac{\partial un}{\partial x1} & \frac{\partial un}{\partial x2} \cdots\cdots & \frac{\partial un}{\partial xn} \end{vmatrix}$$

is called Jacobean of $u_1, u_2, u_3 \cdots\cdots u_n$ with respect to $x_1, x_2, x_3 \cdots\cdots x_n$ and is represented by $\frac{\partial(u_1,u_2,u_3\cdots\cdots u_n)}{\partial(x_1,x_2,x_3\cdots\cdots x_n)}$ or J $(u_1, u_2, u_3 \cdots\cdots u_n)$.

Necessary and sufficient condition for a Jacobian to vanish [1,6,7]

Theorem 3.

Let $u_1, u_2, \dots, u_n$ be function of n independent variables $x_1, x_2, \dots\dots, x_n$. In order that there may exist between these n function a relation,

F $(u_1, u_2, \dots, u_n) = 0$,

It is necessary and sufficient that the Jacobian

$$\frac{\partial\,(u_1, u_2, \dots, u_n)}{\partial\,(x_1, x_2, \dots\dots, x_n)}$$

should vanish identically.

Proof. **The condition is necessary** i.e. if there exists between $u_1, u_2, \dots, u_n$ a relation

$F(u_1, u_2, \cdots\cdots u_n) = 0,$

(i)

there Jacobean is zero. Differentiating (i), we get ,$\frac{\partial F}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_1} + \frac{\partial F}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_1} + \dots + \frac{\partial F}{\partial u_n} \cdot \frac{\partial u_n}{\partial x_1} = 0$ $\frac{\partial F}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_2} + \frac{\partial F}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_2} + \dots + \frac{\partial F}{\partial u_n} \cdot \frac{\partial u_n}{\partial x_2} = 0$

… … … … …

$$\frac{\partial F}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_n} + \frac{\partial F}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_n} + \dots + \frac{\partial F}{\partial u_n} \cdot \frac{\partial u_n}{\partial x_n} = 0.$$

Eliminating $\frac{\partial F}{\partial u_1}, \frac{\partial F}{\partial u_2}, \dots, \frac{\partial F}{\partial u_n}$, *we get* $\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} \cdots & \frac{\partial u_n}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} \cdots & \frac{\partial u_n}{\partial x_2} \\ \frac{\partial u_1}{\partial x_n} & \frac{\partial u_2}{\partial x_n} \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix} = \frac{\partial(u_1, u_2, \cdots\cdots u_2)}{\partial(x_1, x_2, \cdots\cdots, x_n)} = 0$.

Condition is sufficient i.e., if the Jacobean $J(u_1, u_2, \cdots\cdots u_n)$ is zero, the there must exist a relation between $u_1, u_2, \dots, u_n$. The equations connection the functions ($u_1, u_2 \cdots\cdots u_n$) and the variables $x_1, x_2, \dots, x_n$ are always able of being of being transformed into the following forms: $\phi_1(x_1, x_2, \dots, x_n, u_1) = 0$, $\phi_2(x_2, x_3, \dots, x_n,$ $u_1, u_2) = 0$, ⋯ ⋯ ⋯ ⋯

$\phi_r(x_r, x_{r+1}, \dots, x_n, u_1, u_2, \dots, u_r) = 0$, ⋯ ⋯ ⋯ ⋯

$\phi_n(x_n, u_1, u_2 \dots, u_n,) = 0$. Then, we have $J = \frac{\partial(u_1, u_2, \dots\dots, u_n)}{\partial(x_1, x_2, \dots, x_n)} = (-1)^n \frac{\frac{\partial \phi_1}{\partial x_1} \cdot \frac{\partial \phi_2}{\partial x_2} \cdots \frac{\partial \phi_r}{\partial x_r} \cdots \frac{\partial \phi_n}{\partial x_n}}{\frac{\partial \phi_1}{\partial u_1} \cdot \frac{\partial \phi_2}{\partial u_2}, \cdots \frac{\partial \phi_n}{\partial u_n}}$

Now, if $J = 0$, we have $\frac{\partial \phi_1}{\partial x_1} \cdot \frac{\partial \phi_2}{\partial x_2} \cdots \frac{\partial \phi_r}{\partial x_r} \cdots \frac{\partial \phi_n}{\partial x_n} = 0$, i.e. $\frac{\partial \phi_r}{\partial x_r} = 0$,for some value of r between 1 and n. Hence, for that particular value of r the function ϕ_r , must not contain, x_r ; and accordingly the corresponding equation is of the form $\phi_r(x_{r+1},$ $\dots, x_n, u_1, u_2 \dots, u_r) = 0$.

Consequently, between this and the remaining equations $\phi_{r+1} = 0, \phi_{r+2} = 0, \dots, \phi_n =$ $0,$

The variables $x_{r+1} = 0, x_{r+2}, \dots, x_n$ can be eliminated so as to give a find equation between $x_1, u_2, \dots, u_n$.

Conformal Mapping

A geometrical categorization of complex analytic functions is that they retain angles at non-critical locations. Conformal mapping is the mathematical word for this characteristic. Co formality makes sense for any inner product space, however in practice it is commonly applied to Euclidean space with the conventional dot product. A function f: $R^n \rightarrow R^n$ is called conformal if it preserves angles. But what exactly does "preserve angles" mean? The angle between two vectors in the Euclidean norm is defined by their dot product. However, because most analytic maps are nonlinear, they will not map vectors to vectors and will instead map straight lines to curves. However, if we define "angle" as the angle formed by two curves. As a result, complicated functions must be realized as conformal maps. Isogonal mappings retain the magnitude of angles but not necessarily the meaning. A mapping which preserves the magnitude of angles but not necessarily the sense is called isogonal.

Sufficient conditions for w = f(z) represent a conformal mapping

Theorem4. Suppose f(z) be an analytic function of z in a region D of the z-plane and let f'(z) $\neq 0$ inside D. Then the mapping w = f(z) is conformal at the points of D.

Proof: Let z_0 be an interior point of the region D and let C_1 and C_2 be two continuous curves passing through Z_0 making angles $a_1 and\ a_2$ respectively with the real axis. Taking the point Z_1 and Z_2 on the curves C_1 and C_2 at the same distance r from the point $Z_0$where r is small. Then we can write [1,6,7]

$$Z_1 - Z_0 = re^{i\theta}{}_1 \text{and } z_2 - z_0 = re^{i\theta}{}_2.$$

$$\text{As r} \longrightarrow 0,\ \theta_1 \longrightarrow a_1 and\ \theta_2 \longrightarrow a_2.$$

Now as a point moves fromZ_0 and Z_1along C_1, the image point moves along Γ_1 in the w-

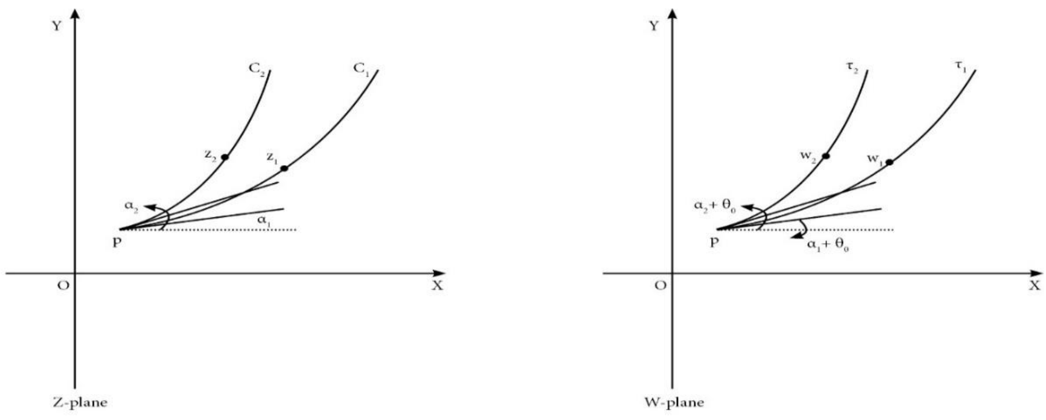

Fig. 5.1 Z-plane and W-plane

plane $w_0\ to\ w_1$. Similarly, as a point moves from$Z_0\ and\ Z_2$along C_2, the image point moves along $\Gamma_2$from $W_0\ to W_2$. Consider that

$$w_1 - w_0 = p_1 e^{i\phi}{}_1, and\ w_2 - w_0 = p_2 e^{i\phi}{}_2 .$$

Since f(z) is analytic, we have

$$\lim_{Z_1 \to Z_0} \frac{W_1 - W_0}{Z_1 - Z_0} = f'(z_0)$$

As f' $(z_0) \neq 0$, we may write$f'(z_0) = R_0 e^{i\phi}{}_0$.

Necessary conditions for w = f(z) to represent a conformal mapping [2]

Theorem5. If w = f(z) represent a conformal transformation of a domain D in the z-plane into a domain D of the w –plane then f(z) is an analytic function of z in D.

Proof. Now, u+ iv = u(x, y) + iv(x, y) then u = u(x, y) and v= v(x, y)

Let ds and dα denote elementary arc lengths in the z-plane and w-plane respectively. Then d s^2= d x^2 + d y^2 and d α^2= d u^2 +d v^2.

du = $\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$ and dv = $\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

Now, dα^2 = $\left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy\right)^2 + \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy\right)^2$

Or, dα^2= E dx^2 + 2F dxdy+ Gdy^2

Where E = $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$, F = $\frac{\partial u}{\partial x} . \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} . \frac{\partial v}{\partial y}$.

G = $\left(\frac{\partial u}{\partial Y}\right)^2 + \left(\frac{\partial v}{\partial Y}\right)^2$

Now, : d s is independent of direction if

$\frac{E}{1} = \frac{F}{0} = \frac{G}{1} = (h)^2$ (suppose) ,

where h depends on x and y only and is not zero. So, the conditions for an isogonal transformation are $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = (h)^2 = \left(\frac{\partial u}{\partial Y}\right)^2 + \left(\frac{\partial v}{\partial Y}\right)^2$

(16)

$\frac{\partial u}{\partial x} . \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} . \frac{\partial v}{\partial y} = 0$

(17)

The equation (16) are satisfied if we arranged

$\frac{\partial u}{\partial x}$ = h $\cos\alpha$, $\frac{\partial v}{\partial x}$ = h $\sin\alpha$, $\frac{\partial u}{\partial y}$ = h $\sin\beta$ and $\frac{\partial v}{\partial y}$ = h $\sin\beta$

Then substituting these values in (17),

$(h)^2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$ =0

$\cos(\alpha-\beta)$ = 0, where h $\neq 0$. Now taking $\alpha-\beta=\frac{\pi}{2}$, i.e. $\alpha = \beta+\frac{\pi}{2}$

$\frac{\partial u}{\partial x}$= h$\cos\left(\frac{\pi}{2}+\beta\right)$ = - h sinβ, $\frac{\partial v}{\partial x}$ = h $\sin\left(\frac{\pi}{2}+\beta\right)$ = - h cosβ.

$\frac{\partial u}{\partial y}$= hcos$\beta$, $\frac{\partial v}{\partial y}$ = h sinβ.

So, $\frac{\partial u}{\partial x}$ = - $\frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x}$ = $\frac{\partial u}{\partial y}$

(18)

Similarly taking $\alpha-\beta=-\frac{\pi}{2}$, i.e. $\alpha = \beta-\frac{\pi}{2}$

$\frac{\partial u}{\partial x}$ = $\frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x}$ = - $\frac{\partial u}{\partial y}$

(19)

The equations (19) are the well-known Cauchy-Riemann equations, which demonstrate that f(z) is an analysis function of z. If we substitute -v for v, that is, if we take the image figure formed by the reflection in the real axis of the w-plane, the equations (18) simplify to (19). Thus, if equation (18) corresponds to an isogonal but not a conformal transformation w = f(z), then f(z) must be an analytic function of z.

Conclusion

A transformation is said to isogonal if two curves in the z – plane intersecting at the point z_o at an angle θ are transformed into two corresponding curves in the w – plane intersecting at the point w_o which corresponds to the point z_0 at the same angleθ . If the sense of the rotation as well as magnitude of the angle is preserved, then the transformation is called conformal. In this case, the magnitude of the angles of a transformation is conserved but their sign is changed. In this case, the magnitude of the angles of a transformation is conserved but their sign is changed. For example, consider the transformation w = x – iy and z = x + iy Therefore, w = x – iy is the reflection of z in the real axis where angles are conserved but their signs are changed.

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