Mikailalsys Journal of
Advanced Engineering International

e-ISSN : 3046-9694
p-ISSN : 3046-8914
Index : Harvard, Boston, Sydney University, Dimensions, Scilit, Semantic, Google, etc

https://doi.org/10.58578/MJAEI.v3i3.11136

Coefficient Estimates and Fekete–Szegö Inequalities for a Class of Pseudo-Starlike P Valent Functions Associated with Chebyshev Polynomials

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Submitted:	Revised:	Accepted:	Published:
Sep 24, 2026	Oct 23, 2026	Nov 3, 2026	Nov 8, 2026

Abstract

This study introduces and investigates a new subclass of p-valent analytic functions in the open unit disk $\mathbb{D}$, combining λ-pseudo-starlikeness with subordination involving Chebyshev polynomials. The analysis derives sharp upper bounds for the initial Taylor–Maclaurin coefficients $|a_{p+1}|$ and $|a_{p+2}|$. It also establishes a Fekete–Szegö inequality for the functional $|a_{p+2} - \mu a_{p+1}^2|$. Several consequences and special cases of the main results are discussed, providing further coefficient and functional estimates for the proposed class.

Keywords: p-Valent Analytic Functions; λ-Pseudo-Starlike Functions; Chebyshev Polynomials; Coefficient Estimates; Fekete–Szegö Inequality

INTRODUCTION

Let $\mathcal{H}$ denote the class of all functions f analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$. A function f is said to be p-valent (or multivalent) if, for each value $w \in \mathbb{C}$, the equation $f(z) = w$ has at most p roots in $\mathbb{D}$, and there exists some $w0$ for which it has exactly p roots [Dubinin, 2015]. We denote by Ap the class of functions f analytic in $\mathbb{D}$ of the form

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$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \qquad p\epsilon\mathbb{N} = \{1,2,3,\dots\} \tag{1.1}$$

For $p = 1$, this reduces to the well-known class of normalized univalent functions.

The study of p-valent functions has been a significant area of geometric function theory. Coefficient estimates, distortion theorems, and Fekete–Szegö inequalities for various subclasses of p-valent functions have been extensively investigated [Aouf, 1987a, b].

Chebyshev polynomials have recently found important applications in geometric function theory, particularly in obtaining sharp coefficient bounds for subclasses of analytic and univalent functions [Altınkaya & Yalçın, 2018]. The Chebyshev polynomials of the second kind *Tn*(*x*) are special cases of Gegenbauer polynomials $C_n^{\gamma}(x)$ when γ=1 [Al-Rawashdeh, 2023]. They satisfy the generating function

$$\frac{1}{1 - 2xz + z^2} = \sum_{n=0}^{\infty} T_n(x) z^n, \tag{1.2}$$

with initial values

$$T_0(x) = 1, \qquad T_1(x) = 2x, \qquad T_2(x) = 4x^2 - 1. \tag{1.3}$$

λ-Pseudo-starlike functions represent a generalization of starlike functions incorporating a rotation parameter *λ*. Recent works have established coefficient estimates and Toeplitz determinant bounds for families of λ-pseudo-starlike functions [Caglar, et al, 2024 & Wanas, et al, 2024]. The connection between p-valent functions and orthogonal polynomials, including Chebyshev polynomials, has been explored through Gegenbauer polynomial subordination [Al-Rawashdeh, (2023), & Al-Rawashdeh, (2024)].

The Fekete–Szegö inequality for the functional $|a_3 - \mu a_2^2|$ (or its p-valent analog $|a_{p+2} - \mu a_{p+1}^2|$) is a classical problem in coefficient theory, providing refined estimates that generalize the second coefficient bound [Fekete & Szegö, 1933].

We are motivated by these developments, we introduce a new subclass of p-valent functions that combines λ-pseudo-starlikeness with Chebyshev polynomial subordination, and investigate its coefficient properties.

METHODS PRELIMINARIES AND DEFINITION

Subordination Principle

For two analytic functions f and g in $\mathbb{D}$, we say that f is subordinate to g, denoted $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$ analytic in $\mathbb{D}$ with $w(0) = 0$ and $|\, w(z) \,| < 1$ such that $f(z) = g(w(z))$ [Miller & Mocanu, 2000]. If g is univalent, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

Definition of the New Class

Definition 2.1. Let $\lambda \in [0,1]$, $0 \leq \alpha < 1, and\ -1 \leq x \leq 1$. A function $f \in Ap$ of the form (1.1) is said to belong to the class $S^{\alpha}_{p,\lambda}(x)$ if it satisfies the following subordination condition:

$$\frac{1}{p}\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)}\right) \prec \frac{1}{1 - 2xz + z^2}, \qquad (2.1)$$

or equivalently,

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} \prec \frac{p}{1 - 2xz + z^2}, \qquad (2.2)$$

where $z \in \mathbb{D}$ and the denominator is non-zero.

Remark 2.1. The function $\frac{1}{1-2xz+z^2}$ is the generating function of the Chebyshev polynomials of the second kind, and it is univalent in $\mathbb{D}$ for $x \in [-1,1]$. Its Maclaurin series is given by

$$\frac{1}{1 - 2xz + z^2} = \sum_{n=0}^{\infty} T_n(x) z^n = 1 + 2xz + (4x^2 - 1)z^2 + \cdots \qquad (2.3)$$

Remark 2.2. For $p = 1$, Definition 2.1 reduces to a class of λ-pseudo-starlike functions associated with Chebyshev polynomials, extending the classes studied in [Caglar, et al, 2024 & Wanas, et al, 2024].

Remark 2.3. The parameter λ controls the type of starlikeness:

- When $\lambda = 0$, the condition reduces to the standard starlike subordination involving Chebyshev polynomials.

- When $\lambda = 1$, the condition involves the operator $zf'(z) + z^2 f''(z)$, related to convexity.

RESULTS

Coefficient Bounds for a_{p+1} and a_{p+2}

Theorem 3.1. Let $f \epsilon\, p_{p,\lambda}^{\alpha}(x)$ be of the form (1.1). Then

$$\left|a_{p+1}\right| \leq \frac{2p|x|\left(1+\lambda(p-1)\right)}{1+p\lambda} \tag{3.1}$$

And

$$\left|a_{p+2}\right| \leq \frac{|4x^2-1|}{|A_2-B_2|} + \left|\frac{B_1^2 - A_1 B_1}{A_2 - B_2}\right| \frac{4P^2x^2\left(1+\lambda(p-1)\right)^2}{(1+p\lambda)^2} \tag{3.2}$$

Where,

$$A_1 = \frac{(p+1)[1+\lambda p]}{p[1+\lambda(p-1)]}, \qquad A_2 = \frac{(p+2)[1+\lambda(p+1)]}{p[1+\lambda(p-1)]}$$

$$B_1 = \frac{(1-\lambda)+\lambda(p+1)}{1+\lambda(p-1)}, \qquad B_2 = \frac{(1-\lambda)+\lambda(p+2)}{1+\lambda(p-1)}$$

The bound for a_{p+1} is sharp for $x = \pm 1$.

Proof. Let

$$F(z) = \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} \tag{3.3}$$

Since $f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \cdots$, we compute the series expansion of $F(z)$ up to the required order.

We have:

$$f(z) = z^p\left(1 + a_{p+1}z + a_{p+2}z^2 + \cdots\right),$$

$$f'(z) = pz^{p-1} + (p+1)a_{p+1}z^p + (p+2)a_{p+2}z^{p+1} + \cdots,$$

$$f''(z) = p(p-1)z^{p-2} + (p+1)pa_{p+1}z^{p-1} + (p+2)(p+1)a_{p+2}z^p + \cdots,$$

Now,

$$zf'(z) + \lambda z^2 f''(z) = p[1 + \lambda(p-1)]z^p + (p+1)[1 + \lambda p]a_{p+1}z^{p+1}$$

$$+(p+2)[1 + \lambda(p+1)]a_{p+2}z^{p+2} + \cdots \quad (3.4)$$

Also,

$$(1-\lambda)f(z) + \lambda z f'(z) = [1 + \lambda(p-1)]z^p$$

$$+[(1-\lambda) + \lambda(p+1)]a_{p+1}z^{p+1} + [(1-\lambda) + \lambda(p+2)]a_{p+2}z^{p+2} + \cdots \quad (3.5)$$

Therefore,

$$F(z) = P[1 + A_1 a_{p+1}z + A_2 a_{p+2}z^2 + \ldots] \times \left[1 - B_1 a_{p+1}z + \left(B_1^2 a_{p+1}^2 - B_2 a_{p+2}\right)z^2\right]$$

where $A1, A2, B1, B2$ are as defined above.

Thus,

$$F(z) = p + p(A_1 - B_1)a_{p+1}z + p\left[(A_2 - B_2)a_{p+2} + \left({B_1}^2 - A_1 B_1\right){a_{p+1}}^2\right]z^2 + \cdots \quad (3.6)$$

Now, from the subordination condition (2.2), we have

$$F(z) \prec \frac{p}{1 - 2xz + z^2} = p + 2pxz + p(4x^2 - 1)z^2 + \cdots. \quad (3.7)$$

Comparing coefficients in (3.6) and (3.7):

$$p(A_1 - B_1)a_{p+1} = 2px \quad (3.8)$$

This gives

$$\left|a_{p+1}\right| = \frac{2|x|}{|A_1 - B_1|} \quad (3.9)$$

Now compute $A_1 - B_1$:

$$A_1 - B_1 = \frac{(P+1)[1 + \lambda p]}{p[1 + \lambda(p-1)]} - \frac{(1-\lambda) + \lambda(p+1)}{1 + \lambda(p-1)}$$

$$= \frac{1 + p\lambda}{p[1 + \lambda(p-1)]} \quad (3.10)$$

Therefore,

$$\left|a_{p+1}\right| = \frac{2p|x|\left(1 + \lambda(p-1)\right)}{1 + p\lambda} \quad (3.11)$$

For the second coefficient, comparing the z^2 terms:

$$p(A_2 - B_2)a_{p+2} + p\left({B_1}^2 - A_1B_1\right)a_{p+1}^2 = p(4x^2 - 1) \qquad (3.12)$$

Thus,

$$a_{p+2} = \frac{4x^2 - 1 - \left({B_1}^2 - A_1B_1\right)a_{p+1}^2}{A_2 - B_2} \qquad (3.13)$$

Taking absolute values and using (3.11) yields (3.2). This completes the proof.

Fekete–Szegö Inequality

Theorem 3.2. Let $f \epsilon S_{p,\lambda}^{\alpha}(x)$ be of the form (1.1). Then for any real parameter μ,

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{|4x^2 - 1|}{|A_2 - B_2|} + \left|\frac{B_1^2 - A_1B_1}{A_2 - B_2} + \mu\right| \frac{4P^2x^2\left(1 + \lambda(p-1)\right)^2}{(1 + p\lambda)^2} \qquad (3.14)$$

Proof.

From the proof of Theorem 3.1, we established the following coefficient relations by comparing the series expansions of $F(z)$ and the Chebyshev generating function:

For the z-coefficient:

$$p(A_1 - B_1)a_{p+1} = 2px \qquad (3.15)$$

which gives

$$a_{p+1} = \frac{2x}{A_1 - B_1} \qquad (3.16)$$

For the z^2 −coefficient:

$$p(A_2 - B_2)a_{p+2} + p\left({B_1}^2 - A_1B_1\right)a_{p+1}^2 = p(4x^2 - 1) \qquad (3.17)$$

Dividing throughout by p (since $p \neq 0$):

$$(A_2 - B_2)a_{p+2} + \left({B_1}^2 - A_1B_1\right)a_{p+1}^2 = 4x^2 - 1 \qquad (3.18)$$

From equation (3.17):

$$(A_2 - B_2)a_{p+2} = 4x^2 - 1 - +\left({B_1}^2 - A_1B_1\right)a_{p+1}^2 \qquad (3.19)$$

Dividing by $(A_2 - B_2)$:

$$a_{p+2} = \frac{4x^2 - 1}{A_2 - B_2} - \frac{B_1^2 - A_1 B_1}{A_2 - B_2} a_{p+1}^2 \tag{3.20}$$

We want to estimate:

$$a_{p+2} - \mu a_{p+1}^2 \tag{3.21}$$

Substituting (3.21) into (3.22):

$$a_{p+2} - \mu a_{p+1}^2 = \frac{4x^2 - 1}{A_2 - B_2} - \frac{B_1^2 - A_1 B_1}{A_2 - B_2} a_{p+1}^2 - \mu a_{p+1}^2 \tag{3.22}$$

Combining the a_{p+1}^2 terms:

$$a_{p+2} - \mu a_{p+1}^2 = \frac{4x^2 - 1}{A_2 - B_2} - \left(\frac{B_1^2 - A_1 B_1}{A_2 - B_2} + \mu \right) a_{p+1}^2 \tag{3.23}$$

Taking absolute values on both sides:

$$\left| a_{p+2} - \mu a_{p+1}^2 \right| = \left| \frac{4x^2 - 1}{A_2 - B_2} - \left(\frac{B_1^2 - A_1 B_1}{A_2 - B_2} + \mu \right) a_{p+1}^2 \right| \tag{3.24}$$

By the triangle inequality $| A - B | \leq | A | + | B |$:

$$\left| a_{p+2} - \mu a_{p+1}^2 \right| \leq \frac{4x^2 - 1}{A_2 - B_2} + \left| \frac{B_1^2 - A_1 B_1}{A_2 - B_2} + \mu \right| \left| a_{p+1} \right|^2 \tag{3.25}$$

From Theorem 3.1, we have:

$$\left| a_{p+1} \right| \leq \frac{2p|x|\left(1 + \lambda(p-1)\right)}{1 + p\lambda} \tag{3.26}$$

Therefore:

$$\left| a_{p+1} \right|^2 \leq \frac{4p^2 x^2 \left(1 + \lambda(p-1)\right)^2}{(1 + p\lambda)^2} \tag{3.27}$$

Substituting (3.27) into (3.25):

$$\left| a_{p+2} - \mu a_{p+1}^2 \right| \leq \frac{|4x^2 - 1|}{|A_2 - B_2|} + \left| \frac{B_1^2 - A_1 B_1}{A_2 - B_2} + \mu \right| \frac{4P^2 x^2 \left(1 + \lambda(p-1)\right)^2}{(1 + p\lambda)^2} \tag{3.28}$$

This completes the proof.

Hence, the sharpness follows from choosing the extremal function corresponding to equality in the subordination condition.

Corollary 3.1. For $\lambda = 0$, the Fekete–Szegö inequality for the class $S_{p,0}^{\alpha}(x)$becomes

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{|4x^2 - 1|}{p + 2 - p} + \left|\frac{p^2 - p(p+1)}{p + 2 - p} + \mu\right| \frac{4P^2x^2}{p^2} \tag{3.29}$$

Corollary 3.2. For $p = 1$, the inequality reduces to the classical Fekete–Szegö inequality for λ-pseudo-starlike functions associated with Chebyshev polynomials:

$$|a_3 - \mu a_2^2| \leq \frac{|4x^2 - 1|}{A_2 - B_2} + \left|\frac{B_1^2 - A_1B_1}{A_2 - B_2} + \mu\right| \frac{4x^2}{(1+\lambda)^2} \tag{3.30}$$

DISCUSSION

Relationship to Existing Results

The class $S_{p,\lambda}^{\alpha}(x)$generalizes several well-known classes:

1. For $\lambda = 0$: The class reduces to p-valent starlike functions of order αa associated with Chebyshev polynomials. The coefficient estimates obtained extend the results of Al-Rawashdeh (2023a) for Gegenbauer polynomials to the special case of Chebyshev polynomials.
2. For $p = 1$: The class reduces to λ-pseudo-starlike functions associated with Chebyshev polynomials, complementing the Toeplitz determinant results of Caglar et al. (2024) and Wanas et al. (2024).
3. For $\alpha = 0$ and $\lambda = 1$: The condition relates to p-valent convex functions, extending the results of Al-Rawashdeh (2023b) on p-valent close-to-convex functions.

Sharpness of Bounds

The bound is sharp when equality holds in both the triangle inequality and the coefficient bound for $|a_{p+1}|$. This occurs when:

$$\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} = \frac{p}{1 - 2xz + z^2}, \tag{4.1}$$

2. The terms in the triangle inequality are aligned in the same direction, which happens for specific values of x and μ.

The extremal function is given by solving the differential equation:

$$\frac{zf_0'(z) + \lambda z^2 f_0''(z)}{(1-\lambda)f_0(z) + \lambda z f_0'(z)} = \frac{p}{1 - 2xz + z^2}, \tag{4.2}$$

For $x = \pm 1$, the bound for $|a_{p+1}|$ is sharp, and the Fekete–Szegö inequality reduces to a particularly simple form.

The coeffcient bounds in Theorem 3.1 are sharp for $x = \pm 1$. The extremal function is given by

$$f_0(z) = z^p \, exp\left(\int_0^z \frac{p(1 + 2xw + (4x^2 - 1)w^2 + \cdots) - p}{w} dw\right), \tag{4.3}$$

which belongs to the class $S_{p,\lambda}^{\alpha}(x)$ and attains equality in the coefficient estimates.

CONCLUSION

In this paper, we have introduced a new class $S_{p,\lambda}^{\alpha}(x)$ of p-valent analytic functions combining λ-pseudo-starlikeness with Chebyshev polynomial subordination. We have established sharp initial coefficient bounds for $|a_{p+1}|$ and $|a_{p+2}|$, and derived the Fekete–Szegö inequality for the functional $|a_{p+2} - \mu a_{p+1}^2|$. The results generalize several known classes and provide a unified framework for studying p-valent functions associated with orthogonal polynomials.

Future research directions include extending these results to Toeplitz and Hankel determinant bounds for the class $S_{p,\lambda}^{\alpha}(x)$, Generalizations to other orthogonal polynomials (Legendre, Gegenbauer, Hermite),

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