Mikailalsys Journal of
Advanced Engineering International

e-ISSN : 3046-9694
p-ISSN : 3046-8914

Index : Harvard, Boston, Sydney University, Dimensions, Lens, Scilit, Semantic, Google, etc

https://doi.org/10.58578/MJAEI.v3i3.10467

Application of Cauchy's Residue Theorem in Evaluating Real and Trigonometric Integrals

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Article Info:

Submitted:	Revised:	Accepted:	Published:
Jun 18, 2026	Jul 16, 2026	Jul 28, 2026	Aug 2, 2026

Abstract

Evaluating definite real and trigonometric integrals remains a fundamental problem in mathematical analysis, particularly when elementary calculus techniques are insufficient or inefficient. This study aims to examine the application of Cauchy's Residue Theorem as an analytical method for evaluating selected definite real and trigonometric integrals. The analysis transforms the original integrals into appropriate contour integrals in the complex plane, identifies the relevant isolated singularities, computes their residues, and applies the Residue Theorem to derive exact integral values. The results show that this approach provides systematic and exact solutions to integrals that are otherwise difficult to evaluate using conventional real-variable methods. The study concludes that residue theory offers an effective and mathematically elegant framework for definite integral evaluation. These findings reinforce the practical significance of complex analysis and provide a methodological reference for selecting suitable contours and calculating residues in advanced mathematical problem-solving.

Keywords: Complex Analysis; Contour Integration; Definite Integrals; Residue Theorem; Trigonometric Integrals

Introduction

Complex analysis is a branch of mathematics that studies functions of a complex variable. One of its most important tools is Cauchy's Residue Theorem, which provides a powerful technique for evaluating contour integrals. Residue theory has numerous applications in mathematics, physics, and engineering.

Many definite real and trigonometric integrals that are difficult to solve using elementary methods can be evaluated efficiently using residue calculus. The purpose of this paper is to demonstrate the application of Cauchy's Residue Theorem in solving selected real and trigonometric integrals.

Preliminary Concepts

Isolated Singularities

A point z_0 is called an isolated singularity of a function $f(z)$, if $f(z)$ is analytic in a neighborhood of z_0 except at z_0 itself.

Residue

The residue of a function $f(z)$ at an isolated singularity z_0 is the coefficient of $(z-z_0)^{-1}$ in its Laurent series expansion.

Cauchy's Residue Theorem

Theorem:

Let $f(z)$ be analytic inside and on a simple closed contour c except for finitely many isolated singularities $z_1, z_2, \dots z_n$, inside c.

Then

$$\int_c f(z)dz = 2\pi i \sum \operatorname{Res}(f, z_k).$$

Where $\operatorname{Res}(f, z_k)$ denotes the residue of $f(z)$ at z_k.

Application of Cauchy's Residue Theorem to Real and Trigonometric Integrals

This section demonstrates the application of Cauchy's Residue Theorem in evaluating selected real and trigonometric integrals. The procedure consists of transforming each integral into a contour integral, identifying the singularities enclosed by the chosen contour, calculating the corresponding residues, and applying the Residue Theorem to obtain the desired result.

Note: Let c denote the unit circle $|z| = 1$.

Evaluate

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1}.$$

Solution

Define the complex function

$$f(z) = \frac{1}{z^2 + 1}.$$

The simple pole occur at

$$z = \pm i.$$

Choose a semicircular contour c_R in the upper half-plane consisting of the interval $[-R, R]$ and the semicircular arc of radius R.

The only pole enclosed by c_R is

$$z = i.$$

Since the pole is simple,

$$\operatorname{Res}(f, i) = \lim_{z \to i} (z - i) \frac{1}{(z - i)(z + i)} = \frac{1}{2i}.$$

By Cauchy's Residue Theorem,

$$\int_{C_R} f(z)dz = 2\pi i\left(\frac{1}{2i}\right) = \pi .$$

As $R \to \infty$, the contribution from the semicircular arc tends to zero by Jordan's Lemma. Therefore,

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi .$$

Hence,

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi .$$

Evaluate

$$I = \int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1}dx, \qquad a>0.$$

Solution

By cosine function

$$\cos(ax) = \mathrm{Re}\left(e^{iax}\right).$$

Thus consider

$$f(z) = \frac{e^{iaz}}{z^2+1}.$$

The poles occur at

$$z = \pm i .$$

Only $z = i$, lies in the upper half-plane.

The residue at $z = i$ is

$$\mathrm{Re}\,s(f,i) = \lim_{z\to i}(z-i)\frac{e^{iaz}}{(z-i)(z+i)} = \frac{e^{-a}}{2i.}$$

Applying the Residue Theorem,

$$\int_{C_R} \frac{e^{aiz}}{z^2+1}dz = 2\pi i\left(\frac{e^{-a}}{2i}\right) = \pi e^{-a} .$$

Taking the real part yields

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1} dx = \pi e^{-a}, \qquad \forall\, a > 0.$$

Evaluate

$$I = \int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}.$$

Solution

Let $$z = e^{i\theta}.$$

Then $$d\theta = \frac{dz}{iz},$$

And $$\cos\theta = \frac{z+z^{-1}}{2}.$$

By substituting into the integral gives

$$I = \int_c \frac{dz}{5+4\left(z+z^{-1}\right)iz}.$$

We have

$$I = \int_c \frac{dz}{i\left(2z^2+5z+2\right)}.$$

The poles are to be obtain by factoring

$$2z^2+5z+2=0.$$

$$(2z+1)(z+2)=0.$$

Hence,

$$z = -\frac{1}{2}, \qquad z = 2.$$

Only

$$z = -\frac{1}{2},$$

lies inside the unit circle.

Thus the residue is

$$\operatorname{Res}\left(\frac{1}{i(2z+1)(z+2)}, -\frac{1}{2}\right) = \frac{1}{3i}.$$

By applying the Residue theorem,

$$I = 2\pi i\left(\frac{1}{3i}\right) = \frac{2\pi}{3}.$$

Therefore,

$$\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} = \frac{2\pi}{3}.$$

Evaluate

$$I = \int_0^{\infty} \frac{dx}{x^4+1}.$$

Solution

The poles of

$$f(z) = \frac{1}{x^4+1}$$

are given as

$$z^4 = -1.$$

Thus

$$z = e^{\frac{i\pi}{4}}, \qquad e^{\frac{3i\pi}{4}}, \quad e^{\frac{5i\pi}{4}}, \quad e^{\frac{7i\pi}{4}}.$$

The poles in the upper half-plane are

$$z = e^{\frac{i\pi}{4}}, \qquad \textit{and} \quad e^{\frac{3i\pi}{4}}.$$

By computing the residues and applying the Residue Theorem gives

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}.$$

Since the integrand is even as,

$$\int_{0}^{\infty}\frac{dx}{x^4+1}=\frac{\pi}{2\sqrt{2}}.$$

Discussion of Results

The examples demonstrate that Cauchy's Residue Theorem provides a systematic method for evaluating both real and trigonometric integrals. The principal advantage of the method lies in reducing integral evaluation to residue computation at isolated singularities. In each case, the choice of contour and identification of poles lead directly to exact closed form results. These examples highlight the efficiency and elegance of complex variable techniques compared with traditional real variable techniques.

Conclusion

This study has demonstrated the usefulness of Cauchy's Residue Theorem in solving real and trigonometric integrals. The theorem provides a direct and elegant method for evaluating integrals through contour integration and residue computation. The examples considered illustrate the effectiveness of complex analysis in solving problems that arise in mathematical analysis and related fields.

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