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A New Inverse Lomax Weibull-G Family of Distributions with Applications

Jamilu Yunusa Falgore[1], Yahaya Abubakar[2], Sani Ibrahim Doguwa[3], Aminu Suleiman Mohammed[4], Abdussamad Tanko Imam[5]
Ahmadu Bello University, Zaria, Nigeria
jamiluyf@gmail.com; ensiliyu2@yahoo.co.uk

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Abstract

The field of statistics is constantly evolving, and new approaches are being developed to model real-world datasets. Despite this, there are still many significant concerns surrounding real data that remain unresolved by existing approaches. One of the drawbacks of the Inverse Lomax distribution is that it belongs to the inverted family of distributions, which limits its application and makes it unsuitable for some situations. Based on these, a new family of distributions called Inverse Lomax Weibull G (ILWG) based on the Inverse Lomax-G and Weibull-G was proposed in this study. Some statistical properties of the family such as the quantile function, moments, and characteristic function were presented. Exponential distribution was used as a member of this family to demonstrate the applicability of the new family. Some statistical properties of the Inverse Lomax Weibull exponential distribution (ILWED) such as quantile function, moments, and characteristic function were demonstrated. ILWED's shapes can be right skewed and symmetric, as the case maybe. Sample quantiles were presented. A simulation study was also presented to explore the desirable properties of the ILWED. Lastly, an application to three (3) different datasets was demonstrated based on the ILWED.

Keywords: Weibull, Inverse Lomax-G family, Exponential Distribution, New Weibull Inverse Lomax Distribution, Weibull-G

Introduction

The field of statistics is constantly evolving, and new approaches are being developed to model real-world datasets. Despite this, there are still many significant concerns surrounding real data that remain unresolved by existing approaches. It is crucial to recognize these limitations and continue to explore new methods and techniques that can improve our understanding of the underlying patterns in the data ([1], [2], and [3]). In this context, statistical distributions play a crucial role in allowing statisticians and practitioners to make predictions and draw conclusions across a wide range of fields of study. The Inverse Lomax distribution and Weibull distribution are two commonly used distributions that have different interpretations and uses but can be studied jointly in specific contexts to yield more comprehensive insights ([4], [5], [6], [7], and [8]). For instance, if you have data that shows the time it takes for a specific event to occur and it is right-skewed, you should consider using both the Inverse Lomax and Weibull distributions to analyze it thoroughly. By examining the parameters of both distributions, you can obtain a more comprehensive understanding of the underlying events and make better predictions or conclusions about the data ([9], [10], and [11]).

Some of the families of Inverse Lomax distribution include the Inverse Lomax-G by [12], the Inverse Lomax exponentiated-G by [13], the Sine Inverse Lomax G by [14], as well as Truncated Inverse Lomax G by [15]. Moreover, some of the Weibull families of distributions include the Weibull-G by [16], the New Weibull-G by [17], the Extended Weibull-G by [18], the Transmuted Weibull G by [19], the Weighted Weibull-G by [20], as well as Harmonic mixture Weibull-G by [21]. The motivation behind this current paper is to propose a family of distributions that has both the properties of Inverse Lomax and Weibull distributions to model data of such kinds. In this paper, we introduced the Inverse lomax Weibull-G (ILWG) which will be used to fit datasets that exhibit both the properties of the Inverse Lomax distribution and Weibull distribution.

Because of its simplicity and mathematical features, the exponential distribution is a common choice for modelling events with constant failure rates. There are only constant failure rate scenarios that can use the exponential distribution. Data with increasing or decreasing failure rates might not be adequately modelled by it. However, extensions are being developed by researchers to solve its limitations [22]. Some of the extensions of the exponential distribution include the Exponentiated Exponential distribution by [23], the weighted exponential distribution by [24], the transmuted generalized exponential distribution by [25], the Weibull exponentiated exponential distribution by [26], the Weibull exponentiated exponential by [27], the modified weighted exponential distribution by [28], as well as the new weighted exponential distribution by [29].

One of the drawbacks of the Inverse Lomax distribution is that it belongs to the inverted family of distributions, which limits its application and makes it unsuitable for some situations [30]. Though widely used, the Weibull distribution has drawbacks when the underlying failure mechanism does not follow a monotonically growing or decreasing hazard function, which could result in modelling that is not accurate. Comparably, the memoryless quality of the exponential distribution is a downside that might not apply in all real-world situations, potentially resulting in modelling errors, particularly when occurrences are not independent or when hazard rates fluctuate over time.

The formulation of the Inverse Lomax Weibull-G (ILWG)

Falgore and Doguwa [12] proposed the Inverse Lomax-G (IL-G) family based on the T-X generator by [31] for any baseline G(.) distribution. The cumulative density function (CDF) and probability density function (PDF) of IL-G are given as

$$F_{IL}(x;\lambda,\gamma,\Delta)=\left[1+\lambda\left\{\frac{(1-G(x;\mathbf{\Delta}))}{G(x;\mathbf{\Delta})}\right\}\right]^{-\gamma};x>0,\lambda,\gamma,\mathbf{\Delta}>0\,, \tag{1}$$

and

$$f_{IL}(x;\lambda,\gamma,\Delta)=\frac{\theta\gamma\lambda g(x;\Delta)}{[G(x;\mathbf{\Delta})]^{2}}\left[1+\lambda\left[\frac{(1-G(x;\mathbf{\Delta}))}{G(x;\mathbf{\Delta})}\right]\right]^{-(1+\gamma)};x>0,\lambda,\gamma,\mathbf{\Delta}>0\,. \tag{2}$$

Where $\mathbf{\Delta}$ is a vector of parameter(s) for the baseline distribution. The CDF of Weibull-G by

[16] is considered as $G(x;\mathbf{\Delta}) = 1 - e^{-\beta\left(\frac{G(x;\zeta))}{1-G(x;\zeta))}\right)^{\alpha}}$, where ζ is a vector of parameters. Let the scale parameter β assumes value 1 then by considering $G(x;\mathbf{\Delta})$ as the baseline CDF and replacing back in equation (1), we have the CDF and PDF of the ILWG as:

$$F(x;\lambda,\gamma,\alpha,\zeta) = \left[1 + \frac{\lambda e^{-\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}}{1 - e^{-\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}}\right]^{-\gamma} ; x > 0, \lambda, \gamma, \alpha, \zeta > 0 \ , \tag{3}$$

and

$$f(x;\lambda,\gamma,\alpha,\zeta) = \frac{\alpha\gamma\lambda g(x;\zeta)\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha-1} e^{-\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}}{[1-G(x;\zeta)]^{2}\left[1 - e^{-\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}\right]^{2}} \left[1 + \left\{\frac{\lambda e^{-\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}}{1 - e^{-\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}}\right\}\right]^{-(1+\gamma)} ; x > 0, \lambda, \gamma, \alpha, \zeta > 0 \tag{4}$$

The Reliability, Hazard Rate, and Cumulative Hazard Functions of the Inverse Lomax Weibull-G

R(t) represents the reliability function, sometimes referred to as the survival function. It is the likelihood that the unpredictable event (time of failure) will occur after time t. The R(t) of the ILWG can be given as:

$$R(t;\lambda,\gamma,\alpha,\zeta) = 1 - \left[1 + \frac{\lambda e^{-\left[\frac{G(t;\zeta)}{1-G(t;\zeta)}\right]^{\alpha}}}{1 - e^{-\left[\frac{G(t;\zeta)}{1-G(t;\zeta)}\right]^{\alpha}}}\right]^{-\gamma} ; t > 0, \lambda, \gamma, \alpha, \zeta > 0 \tag{5}$$

The hazard function, abbreviated as h(t), is the conditional likelihood that a component will fail in a short period, assuming that it has survived from time zero to the start of the interval. The hazard function of the ILWG family can be given as:

$$h(t)=\frac{f(t;\lambda,\gamma,\alpha,\zeta)}{R(t;\lambda,\gamma,\alpha,\zeta)}=\frac{\alpha\gamma\lambda g(t;\zeta)\left[\frac{G(t;\zeta)}{1-G(t;\zeta)}\right]^{\alpha-1}e^{-\left[\frac{G(t;\zeta)}{1-G(t;\zeta)}\right]^{\alpha}}\left[1+\left\{\frac{\lambda e^{-\left[\frac{G(t;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}}{1-e^{-\left[\frac{G(t;\zeta)}{1-G(t;\zeta)}\right]^{\alpha}}}\right\}\right]^{-(1+\gamma)}}{[1-G(t;\zeta)]^{2}\left[1-e^{-\left[\frac{G(t;\zeta)}{1-G(t;\zeta)}\right]^{\alpha}}\right]^{2}\left\{1-\left[1+\frac{\lambda e^{-\left[\frac{G(t;\zeta)}{1-G(t;\zeta)}\right]^{\alpha}}}{1-e^{-\left[\frac{G(t;\zeta)}{1-G(t;\zeta)}\right]^{\alpha}}}\right]^{-\gamma}\right\}} \tag{6}$$

The area under the hazard rate function, represented by the symbol H(t), represents the cumulative hazard rate. Calculating average failure rates is an essential application of this tool. The H(t) of the ILWG is given by:

$$H(t;\lambda,\gamma,\alpha,\zeta)=\int_{t}^{\infty}h(v)dv=-log(R(t;\lambda,\gamma,\alpha,\zeta))=-log\left\{1-\left[1+\frac{\lambda e^{-\left[\frac{G(t;\zeta)}{1-G(t;\zeta)}\right]^{\alpha}}}{1-e^{-\left[\frac{G(t;\zeta)}{1-G(t;\zeta)}\right]^{\alpha}}}\right]^{-\gamma}\right\} \tag{7}$$

The Statistical Properties of the ILWG Family of Distributions

Here, we derive some of the mathematical properties of the ILWG.

Linear Representation of the ILWG's Density and Distribution functions

Let $(1+v)^{-t}=\sum_{k_1=0}^{\infty}\frac{(-1)^{k_1}\Gamma(t+k_1)}{\Gamma(t)k_1!}v^{k_1}$ be a binomial expansion. Then, by using the relations,

$$\left[1+\left[\frac{\lambda e^{-\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}}{1-e^{-\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}}\right]\right]^{-\gamma}=\sum_{i_1=0}^{\infty}\frac{(-1)^{i_1}\Gamma(\gamma+i_1)}{\Gamma(\gamma)i_1!}\left[\frac{\lambda e^{-\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}}{1-e^{-\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}}\right]^{i_1} \tag{8}$$

And

$$\left[\frac{\lambda e^{-\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}}{1-e^{-\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}}\right]^{i_1}=\lambda^{i_1}e^{-i_1\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}\left[1-e^{-\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}\right]^{-i_1} \tag{9}$$

Also,

$$\left[1-e^{-\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}\right]^{-i_1}=\sum_{i_2=0}^{\infty}\frac{\Gamma(i_1+i_2)}{\Gamma(i_1)i_2!}e^{-i_2\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}} \tag{10}$$

and

$$e^{-(i_1+i_2)\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}=\sum_{i_3=0}^{\infty}\frac{(-1)^{i_3}(i_1+i_2)^{i_3}\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha i_3}}{i_3!} \tag{11}$$

$$[1-G(x;\zeta)]^{-\alpha i_3}=\sum_{i_4=0}^{\infty}\frac{\Gamma(\alpha i_3+i_4)}{i_4!\Gamma(\alpha i_3)}G(x;\zeta)^{i_4}\, then, \left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha i_3}=\sum_{i_4=0}^{\infty}\frac{\Gamma(\alpha i_3+i_4)}{i_4!\Gamma(\alpha i_3)}G(x;\zeta)^{\alpha i_3+i_4} \tag{12}$$

Therefore, the CDF of the ILWG can be re-written as:

$$F(x;\lambda,\gamma,\alpha,\zeta)=\left[1+\frac{\lambda e^{-\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}}{1-e^{-\left[\frac{G(x;\zeta)}{1-G(x;\zeta)}\right]^{\alpha}}}\right]^{-\gamma}=\sum_{i_1,i_2,i_3,i_4=0}^{\infty}\Omega_{(\alpha i_3+i_4)}H_{(\alpha i_3+i_4)}(x;\zeta)$$

(13) where,

$$\Omega_{(\alpha i_3+i_4)}=\frac{\lambda^{i_1}(i_1+i_2)^{i_3}(-1)^{i_2+i_3}\frac{(-1)^{i_1}\Gamma(t+k_1)}{\Gamma(t)i_1!}\frac{\Gamma(i_1+i_2)}{\Gamma(i_1)i_2!}\Gamma(\alpha i_3+i_4)}{i_4!\Gamma(\alpha i_3)}$$, and

$H_{(\alpha i_3+i_4)}(x;\zeta)=G^{(\alpha i_3+i_4)}(x;\zeta).$ The PDF corresponding to equation (\ref{328}) is given by:

$$f(x;\lambda,\gamma,\alpha,\zeta)=\sum_{i_1,i_2,i_3,i_4=0}^{\infty}\Omega_{(\alpha i_3+i_4)}h_{(\alpha i_3+i_4-1)}(x;\zeta) \tag{14}$$

Where $h_{(\alpha i_3+i_4-1)}(x;\zeta)=(\alpha i_3+i_4)g(x;\zeta)G(x;\zeta)^{(\alpha i_3+i_4)-1}$ and (αi_3+i_4) is the power parameter

The quantile function of the ILWG family of Distributions

The quantile function of ILWG is given as:

$$Q(U) = G(x;\zeta)^{-1}\left[\frac{\left[\frac{U^{\frac{-1}{\gamma}}-1}{U^{\frac{-1}{\gamma}}+\lambda-1}\right]^{\frac{1}{\alpha}}}{1-\left[\frac{U^{\frac{-1}{\gamma}}-1}{U^{\frac{-1}{\gamma}}+\lambda-1}\right]^{\frac{1}{\alpha}}}\right] \tag{15}$$

Where U is uniformly distributed between 0 and 1. The median of the ILWG family can be derived by setting U=0.5 in equation (15) as:

$$Q(0.5) = G(x;\zeta)^{-1}\left[\frac{\left[\frac{(\frac{1}{2})^{\frac{-1}{\gamma}}-1}{(\frac{1}{2})^{\frac{-1}{\gamma}}+\lambda-1}\right]^{\frac{1}{\alpha}}}{1-\left[\frac{(\frac{1}{2})^{\frac{-1}{\gamma}}-1}{(\frac{1}{2})^{\frac{-1}{\gamma}}+\lambda-1}\right]^{\frac{1}{\alpha}}}\right] \tag{16}$$

Where $G(x;\zeta)^{-1}$ is a quantile function of the baseline distribution.

Moments of the ILWG family of Distributions

Let X be a random variable that follows ILWG with parameters $(\lambda,\gamma,\alpha,\zeta)$, then the d^{th} moment about the origin is given by:

$$\mu_d' = E(X^d) = \int_{-\infty}^{\infty} x^d f(x;\lambda,\gamma,\alpha,\zeta)dx \tag{17}$$

Using some of the results under linear representation of the ILWG, we have the moment of the ILWG as:

$$\mu_d' = \sum_{i_1,i_2,i_3,i_4=0}^{\infty} \Omega_{(\alpha i_3+i_4)} \int_0^{\infty} x^d h_{(\alpha i_3+i_4-1)} dx \tag{18}$$

The mean of the ILWG can be derived by setting d=1 in equation (18). In the same vein, the second moment can also be derived by setting d=2, and then using the relation:

$$Var(X)=\mu_2'-[\mu_1']^2 \tag{19}$$

To find the variance.

The Characteristic and Moment Generating Functions of the ILWG Family

The characteristic function of the ILWG can be given as:

$$\phi(t)=\int_0^\infty e^{itx}f(x;\lambda,\gamma,\alpha,\zeta)dx=\sum_{i_1,i_2,i_3,i_4=0}^{\infty}\Omega_{(\alpha i_3+i_4)}\int_0^\infty e^{itx}h_{(\alpha i_3+i_4-1)}dx \tag{20}$$

And the moment-generating function of the ILWG can be given as:

$$M_X(t)=\int_0^\infty e^{tx}f(x;\lambda,\gamma,\alpha,\zeta)dx=\sum_{i_1,i_2,i_3,i_4=0}^{\infty}\Omega_{(\alpha i_3+i_4)}\int_0^\infty e^{tx}h_{(\alpha i_3+i_4-1)}dx \tag{21}$$

The Maximum Likelihood Estimation (MLE) of the parameters of the ILWG

In this section, we used the maximum likelihood method to estimate the parameters of the ILWG family of distributions. The Maximum Likelihood approach has the feature of ensuring MLEs with interesting properties such as asymptotic unbiasedness and normalcy. The asymptotic unbiasedness, in particular, provides theoretical guarantees that, for sample size (n) high enough, the MLEs must be near the actual unknown parameter values. There is, however, no definite guarantee for very small n. Let $x_1,x_2,x_3,\dots,x_n.$ be a random sample independently drawn from ILWG family. Then, the log-likelihood function $L(\lambda,\gamma,\alpha,\zeta)$ of equation (4) is given as:

$$\begin{aligned}L(x;\lambda,\gamma,\alpha,\zeta)&=nlog(\lambda\gamma\alpha)+\sum_{i=1}^{n}log(g(x_i;\zeta))+(\alpha-1)\sum_{i=1}^{n}log\left(W(x_i;\zeta)\right)-\sum_{i=1}^{n}W(x_i;\zeta)^{\alpha}\\&-2\sum_{i=1}^{n}log(1-G(x_i;\zeta))-2\sum_{i=1}^{n}log\left(1-e^{-W(x_i;\zeta)^{\alpha}}\right)-(1+\gamma)\sum_{i=1}^{n}log\left(1+\left[\frac{\lambda e^{-W(x_i;\zeta)^{\alpha}}}{\left[1-e^{-W(x_i;\zeta)^{\alpha}}\right]}\right]\right)\end{aligned} \tag{22}$$

Where $W(x_i;\zeta)=\left(\frac{G(x_i;\zeta)}{1-G(x_i;\zeta)}\right)$. Taking the partial derivatives of equation (22) with respect to $\gamma,\lambda,\alpha,$ and ζ yields:

$$\frac{\partial L}{\partial \gamma}=\frac{n}{\gamma}-\sum_{i=1}^{n} log\left(1+\left[\frac{\lambda e^{-W(x_i;\zeta)^{\alpha}}}{\left[1-e^{-W(x_i;\zeta)^{\alpha}}\right]}\right]\right) \tag{23}$$

$$\frac{\partial L}{\partial \lambda}=\frac{n}{\lambda}-(1+\gamma)\sum_{i=1}^{n}\frac{e^{-W(x_i;\zeta)^{\alpha}}}{\left[1+\left\{\frac{\lambda e^{-W(x_i;\zeta)^{\alpha}}}{1-e^{-W(x_i;\zeta)^{\alpha}}}\right\}\right]\left[1-e^{-W(x_i;\zeta)^{\alpha}}\right]} \tag{24}$$

$$\begin{aligned}\frac{\partial L}{\partial \alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}W(x_i;\zeta)^{\alpha} log(W(x_i;\zeta))-2\sum_{i=1}^{n}\frac{W(x_i;\zeta)^{\alpha} log(W(x_i;\zeta))e^{-W(x_i;\zeta)^{\alpha}}}{\left[1-e^{-W(x_i;\zeta)^{\alpha}}\right]}\\ +\sum_{i=1}^{n} log\left(W(x_i;\zeta)\right)-(1+\gamma)\sum_{i=1}^{n}\frac{\lambda W(x_i;\zeta)^{\alpha} log(W(x_i;\zeta)^{\alpha})e^{-W(x_i;\zeta)^{\alpha}}}{\left[1-e^{-W(x_i;\zeta)^{\alpha}}\right]^2\left[1+\left(\frac{\lambda e^{-W(x_i;\zeta)^{\alpha}}}{\left\{1-e^{-W(x_i;\zeta)^{\alpha}}\right\}}\right)\right]}\end{aligned} \tag{25}$$

$$\begin{aligned}\frac{\partial L}{\partial \zeta}=\sum_{i=1}^{n}\frac{g'(x_i;\zeta)}{g(x_i;\zeta)}-\sum_{i=1}^{n}\frac{\alpha W(x_i;\zeta)^{\alpha-1}g(x_i;\zeta)}{(1-G(x_i;\zeta))^2}+2\sum_{i=1}^{n}\frac{g(x_i;\zeta)}{\{1-G(x_i;\zeta)\}}+\sum_{i=1}^{n}\frac{(\alpha-1)g(x_i;\zeta)}{W(x_i;\zeta)(1-G(x_i;\zeta))^2}\\ -2\sum_{i=1}^{n}\frac{\alpha W(x_i;\zeta)^{\alpha-1}g(x_i;\zeta)e^{-W(x_i;\zeta)^{\alpha}}}{\left[1-e^{-W(x_i;\zeta)^{\alpha}}\right](1-G(x_i;\zeta))^2}-(1+\gamma)\sum_{i=1}^{n}\frac{\alpha\lambda W(x_i;\zeta)^{\alpha-1}g(x_i;\zeta)e^{-W(x_i;\zeta)^{\alpha}}}{\left[1+\left\{\frac{\lambda e^{-W(x_i;\zeta)^{\alpha}}}{1-e^{-W(x_i;\zeta)^{\alpha}}}\right\}\right]\left[1-e^{-W(x_i;\zeta)^{\alpha}}\right]^2(1-G(x_i;\zeta))^2}\end{aligned} \tag{26}$$

Setting the non-linear equations (23), (24), (25), and (26) to zero and then solving them simultaneously produced the MLE of the parameters $\gamma, \lambda, \alpha,$ and ζ, respectively.

Inverse Lomax Weibull Exponential Distribution (ILWED)

Definition and Graphical presentations of the ILWED

The exponential distribution, also known as the negative exponential distribution in probability theory and statistics, is the probability distribution of the time between events in a Poisson point process, that is, a process in which events occur continuously and independently at a constant average rate. It is a sub-model of the gamma distribution. The CDF and PDF of the ILWED can be given as:

$$F(x;\alpha,\lambda,\gamma,\upsilon)=\left[1+\lambda\left\{\frac{e^{-[e^{\upsilon x}-1]^{\alpha}}}{[1-e^{-[e^{\upsilon x}-1]^{\alpha}}]}\right\}\right]^{-\gamma}, x,\upsilon,\gamma,\lambda,\alpha>0 \tag{27}$$

And

$$f(x;\alpha,\lambda,\gamma,\upsilon)=\frac{\upsilon\alpha\gamma\lambda[e^{\upsilon x}-1]^{\alpha-1}e^{\upsilon x-[e^{\upsilon x}-1]^{\alpha}}\left[1+\lambda\left\{\frac{e^{-[e^{\upsilon x}-1]^{\alpha}}}{(1-e^{-[e^{\upsilon x}-1]^{\alpha}})}\right\}\right]^{-\gamma-1}}{[1-e^{-[e^{\upsilon x}-1]^{\alpha}}]^{2}},\ x,\upsilon,\gamma,\lambda,\alpha>0 \tag{28}$$

Where α and γ are the shape parameters, λ is a scale parameter, and υ is the rate parameter. The reliability and hazard functions of the ILWED are presented in equations (29) and (30).

$$R(x;\lambda,\gamma,\alpha,\upsilon)=\int_{x}^{\infty}f(x;\lambda,\gamma,\alpha,\upsilon)dt=1-\left[1+\lambda\left\{\frac{e^{-[e^{\upsilon x}-1]^{\alpha}}}{[1-e^{-[e^{\upsilon x}-1]^{\alpha}}]}\right\}\right]^{-\gamma};t,\lambda,\gamma,\alpha,\upsilon>0 \tag{29}$$

$$h(x;\lambda,\gamma,\alpha,\upsilon)=\frac{f(x;\lambda,\gamma,\alpha,\upsilon)}{R(x;\lambda,\gamma,\alpha,\upsilon)}=\frac{\upsilon\alpha\gamma\lambda e^{\upsilon x}[e^{\upsilon x}-1]^{\alpha-1}e^{-[e^{\upsilon x}-1]^{\alpha}}\left[1+\lambda\left\{\frac{e^{-[e^{\upsilon x}-1]^{\alpha}}}{(1-e^{-[e^{\upsilon x}-1]^{\alpha}})}\right\}\right]^{-\gamma-1}}{\left\{1-\left[1+\lambda\left\{\frac{e^{-[e^{\upsilon x}-1]^{\alpha}}}{[1-e^{-[e^{\upsilon x}-1]^{\alpha}}]}\right\}\right]^{-\gamma}\right\}[1-e^{-[e^{\upsilon x}-1]^{\alpha}}]^{2}} \tag{30}$$

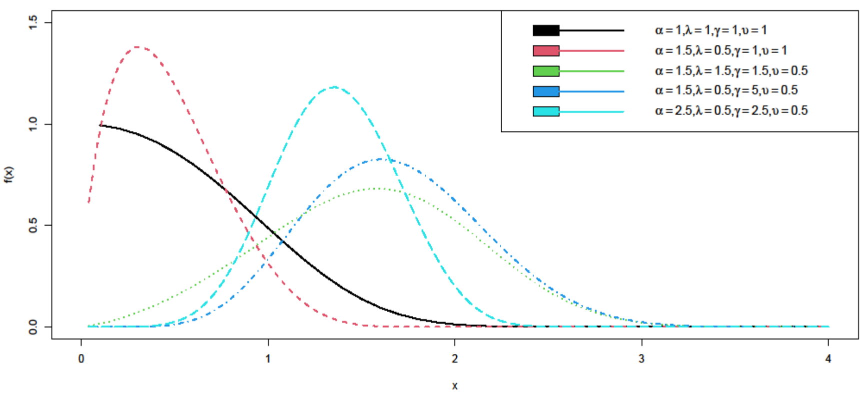

Figure 1: *The PDF of the ILWED at various parameter values*

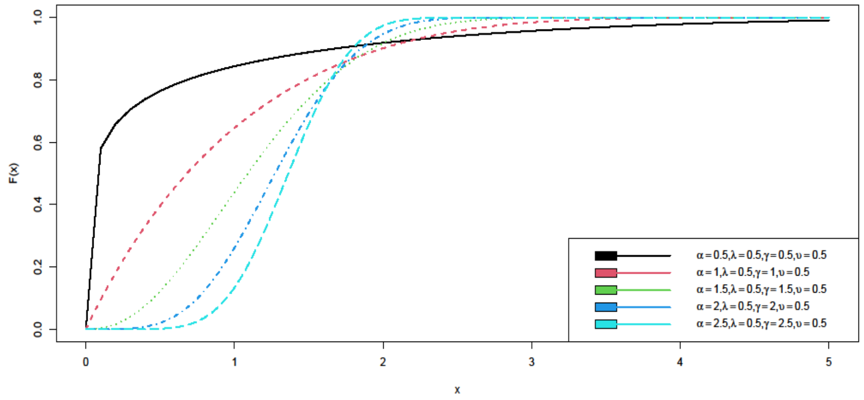

Figure 2: *The CDF of the ILWED at various parameter values*

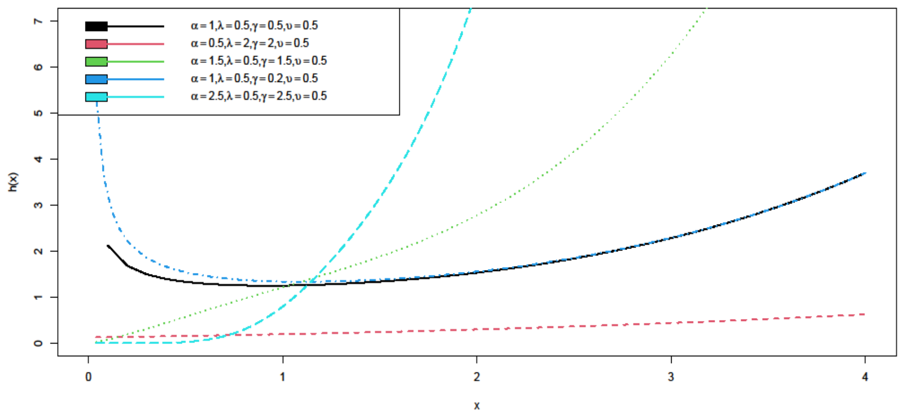

Figure 3: *The Hazard function of the ILWED at various parameter values*

The Simulation Studies of the ILWED

A simulation analysis is carried out and reported here to demonstrate the performance of the estimates at some parameter values. The Monte Carlo method is any computational methodology using pseudo-random values to solve mathematical problems [32]. The numerical study is as follows:

Step 1: For known parameter values i.e $\Theta = (\alpha, \lambda, \gamma, \upsilon)^T$ we simulated a random sample of size n from the ILWED using Equation (15). Step 2: then Estimate the parameters of the ILWED by using MLE. Step 3: Perform 1,000 replications of steps 1 through 2. Step 4: For each of the four (4) parameters of the ILWED, we compute the Biases and Root Mean Squared Errors (RMSEs) of the parameters from the 1,000 parameter estimates. The sample sizes considered were (n=10, 20, 30, 50, 70, 90, 150, and 170). The statistics are given by

$$\hat{\Theta}=\frac{1}{1,000}\sum_{i=1}^{1,000}\Theta_i,\ Bias(\hat{\Theta})=\frac{1}{1,000}\sum_{i=1}^{1,000}(\Theta_i-\Theta),\ RMSE(\hat{\Theta})=\sqrt{\frac{1}{1,000}\sum_{i=1}^{1,000}(\Theta_i-\Theta)^2} \quad (31)$$

Where $\Theta_i=(\hat{\alpha},\hat{\upsilon},\hat{\gamma},\hat{\lambda})$ is the MLE for each iteration (n=10, 20, 30, 50, 70, 90, 150, 170). Three cases were considered for the simulation. Case I:(α = 0.7, λ = 0.5, γ = 0.5 and υ = 0.05), Case II:(α = 1, λ = 0.5, γ = 0.5 and υ = 0.5), and Case III: (α = 1, λ = 0.1, γ = 0.5 and υ = 0.001). Tables (1), (2), and (3) are for Case I, Case II and Case III, respectively.

Table 1: *Simulation Results for Case I*

n	Estimates	Bias	RMSE	n	Estimates	Bias	RMSE
10	0.9397	0.2397	0.3589	70	0.8601	0.1601	0.2004
	0.7695	0.2695	0.5455		0.6877	0.1877	0.3181
	0.7300	0.2300	0.3457		0.6395	0.1395	0.1861
	0.0683	0.0183	0.0417		0.0523	0.0023	0.0105
20	0.9067	0.2067	0.2945	90	0.8560	0.1560	0.1881
	0.7169	0.2169	0.4460		0.6822	0.1822	0.2964
	0.6807	0.1807	0.2792		0.6327	0.1327	0.1700
	0.0580	0.0080	0.0219		0.0517	0.0017	0.0086
30	0.8853	0.1853	0.2531	150	0.8458	0.1458	0.1702
	0.7141	0.2141	0.4186		0.6881	0.1881	0.2704
	0.6557	0.1557	0.2313		0.6286	0.1286	0.1563
	0.0549	0.0049	0.0163		0.0516	0.0016	0.0069
50	0.8691	0.1691	0.2227	170	0.8424	0.1424	0.1620
	0.6922	0.1922	0.3538		0.6874	0.1874	0.2617
	0.6463	0.1463	0.2063		0.6282	0.1282	0.1509
	0.0531	0.0032	0.0124		0.0516	0.0016	0.0068

Table 2: *Simulation Results for Case II*

n	Estimates	Bias	RMSE	n	Estimates	Bias	RMSE
10	1.1941	0.1941	0.4818	70	1.0207	0.0207	0.1979
	0.6737	0.1737	0.5850		0.5482	0.0482	0.2542
	0.6156	0.1156	0.3448		0.5375	0.0375	0.1655
	0.6534	0.1534	0.2967		0.5329	0.0329	0.1067
20	1.1091	0.1091	0.3586	90	1.0096	0.0096	0.1698
	0.6287	0.1287	0.4867		0.5326	0.0326	0.2151
	0.5723	0.0723	0.2564		0.5325	0.0325	0.1400
	0.5927	0.0927	0.2113		0.5248	0.0248	0.0910
30	1.0729	0.0729	0.2990	150	1.0058	0.0058	0.1350
	0.6165	0.1165	0.4321		0.5251	0.0251	0.1723
	0.5511	0.0511	0.2220		0.5200	0.0200	0.1135
	0.5666	0.0666	0.1772		0.5149	0.0149	0.0706
50	1.0351	0.0351	0.2385	170	0.9982	-0.0018	0.1221
	0.5598	0.0598	0.3007		0.5225	0.0225	0.1538
	05465	0.0465	0.1924		0.5178	0.0178	0.0925
	0.5424	0.0424	0.1253		0.5157	0.0157	0.0651

Table 3: *Simulation Results for Case III*

n	Estimates	Bias	RMSE	n	Estimates	Bias	RMSE
10	1.0772	0.0772	0.1889	70	1.0152	0.0152	0.0445
	0.2033	0.1033	0.1566		0.1171	0.0171	0.0400
	0.5416	0.0416	0.1909		0.4987	-0.0013	0.0491
	0.0028	0.0018	0.0037		0.0012	0.0002	0.0004
20	1.0425	0.0425	0.1082	90	1.0129	0.0129	0.0389
	0.1524	0.0524	0.0926		0.1125	0.0125	0.0332
	0.5072	0.0072	0.1021		0.4998	-0.0002	0.0436
	0.0017	0.0007	0.0013		0.0011	0.0001	0.0003
30	1.0278	0.0278	0.0742	150	1.0107	0.0107	0.0313
	0.1408	0.0408	0.0751		0.1077	0.0077	0.0241
	0.4976	-0.0024	0.0764		0.4992	-0.0008	0.0344
	0.0015	0.0005	0.0008		0.0011	0.0001	0.0002
50	1.0174	0.0174	0.0573	170	1.0088	0.0088	0.0280
	0.1240	0.0240	0.0507		0.1070	0.0070	0.0225
	0.4994	-0.0006	0.0587		0.5005	0.0005	0.0318
	0.0013	0.0003	0.0005		0.0011	0.0001	0.0002

As the value of the sample size (n) increases, the simulation results of the ILWED show:

- Stability of the MLES,
- The bias of the MLEs approach zero, and
- Decrease in the RMSEs of the MLEs.

Applications of the ILWED to Three Datasets

Inverse Lomax Weibull Exponential Distribution (ILWED) was fitted to three datasets. This include datasets with increasing and bathtub hazard shapes. The Goodness-of-fit criteria used are the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). Moreover, three Goodness-of-fir statistics were used. These are the Kolmogorov-Smirnov (K-s), Anderson-Darling (A-D), and Cramer-Von Mises (C-vM). By comparing the cumulative distribution functions (PDFs) of two datasets, the Kolmogorov-Smirnov test determines whether they have the same continuous distribution. It is a widely applicable test that helps to compare theoretical and empirical distributions and is especially helpful when parametric assumptions are not met. The A-D test is useful for determining fit, especially in situations with extreme values or interesting tail behavior. It does this by calculating the goodness-of-fit between the sample's empirical distribution function and the specified distribution's cumulative distribution function. This test is sensitive to deviations in the distribution's tails. The C-vM test, named for Carl von Mises and Harald Cramer, measures the difference between the sample's empirical distribution function and the specified distribution's cumulative distribution function to determine whether the sample fits the distribution. This test is favored in some circumstances because it is easy to compute and provides a good measure of goodness-of-fit. Finally, The negative log-likelihood (-ll), which is frequently minimized in maximum likelihood estimation, measures how well a model fits observed data by estimating the probability of observing the data given the model parameters. It is favored for its stability and ease of use in parameter estimation and is essential to many different disciplines, including biology, econometrics, and machine learning. ILWED was fitted alongside the Weibull Inverse Lomax Distribution by (WIL) by [10], Transmuted Generalized Exponential Distribution (TGED) by Khan [25], Alpa Power Exponential Distribution (APED) by [33], the four parameters Weibull Exponentiated

Exponential Distribution (WEED) by [27], another three parameters Weibull Exponentiated Exponential Distribution (WExED) by [26], the Inverse Exponentiated Odd Lomax Exponential Distribution (IEOLE) by [34], the Extended Odd Frechet Weibull Distribution (EOFWD) by [35], Weibull Distribution (WD), as well as the Exponential Distribution (ED).

Application to the Italy Covid-19 Dataset

These data was reported by the [36], belonging to Italy Covid-19 patients for 172 days, from 1 March to 21 August 2020.

0.0490, 0.0601, 0.0460, 0.0533, 0.0630, 0.0297, 0.0885, 0.0540, 0.1720, 0.0847, 0.0713, 0.0989, 0.0495, 0.1025, 0.1079, 0.0984, 0.1124, 0.0807, 0.1044, 0.1212, 0.1167, 0.1255, 0.1416, 0.1315, 0.1073, 0.1629, 0.1485, 0.1453, 0.2000, 0.2070, 0.1520, 0.1628, 0.1666, 0.1417, 0.1221, 0.1767, 0.1987, 0.1408, 0.1456, 0.1443, 0.1319, 0.1053, 0.1789, 0.2032, 0.2167, 0.1387, 0.1646, 0.1375, 0.1421, 0.2012, 0.1957, 0.1297, 0.1754, 0.1390, 0.1761, 0.1119, 0.1915, 0.1827, 0.1548, 0.1522, 0.1369, 0.2495, 0.1253, 0.1597, 0.2195, 0.2555, 0.1956, 0.1831, 0.1791, 0.2057, 0.2406, 0.1227, 0.2196, 0.2641, 0.3067, 0.1749, 0.2148, 0.2195, 0.1993, 0.2421, 0.2430, 0.1994, 0.1779, 0.0942, 0.3067, 0.1965, 0.2003, 0.1180, 0.1686, 0.2668, 0.2113, 0.3371, 0.1730, 0.2212, 0.4972, 0.1641, 0.2667, 0.2690, 0.2321, 0.2792, 0.3515, 0.1398, 0.3436, 0.2254, 0.1302, 0.0864, 0.1619, 0.1311, 0.1994, 0.3176, 0.1856, 0.1071, 0.1041, 0.1593, 0.0537, 0.1149, 0.1176, 0.0457, 0.1264, 0.0476, 0.1620, 0.1154, 0.1493, 0.0673, 0.0894, 0.0365, 0.0385, 0.2190, 0.0777, 0.0561, 0.0435, 0.0372, 0.0385, 0.0769, 0.1491, 0.0802, 0.0870, 0.0476, 0.0562, 0.0138, 0.0684, 0.1172, 0.0321, 0.0327, 0.0198, 0.0182, 0.0197, 0.0298, 0.0545, 0.0208, 0.0079, 0.0237, 0.0169, 0.0336, 0.0755, 0.0263, 0.0260, 0.0150, 0.0054, 0.0375, 0.0043, 0.0154, 0.0146, 0.0210, 0.0115, 0.0052, 0.2512, 0.0084, 0.0125, 0.0125, 0.0109 , 0.0071.

Table 4: *MLEs and Goodness-of-fit Criteria for the fitted ILWED and other comparators for the comparators for the Italy Covid-19 Dataset*

Distributions	Estimates	Standard Error	ll	AIC	BIC
ILWED$(\upsilon,\alpha,\gamma,\lambda)$	0.2455	0.0471	197.6782	-387.3565	-374.7665
	0.0024	0.0002			
	1411.281	246.2645			
	19.9779	4.4775			
WEED$(\upsilon,\alpha,\gamma,\lambda)$	0.3028	0.2059	196.2795	-384.5591	-371.9691
	12.0909	23.0638			
	123.2842	1510.8660			
	0.0776	0.1323			
WExED(α,γ,λ)	0.2669	0.2064	195.7915	-385.5830	-376.1405
	1.3376	1.9689			
	3.6252	2.2766			

TGED(α,γ,λ)	1.3876	0.2222	190.2449	-374.4898	-365.0474
	11.2037	0.9285			
	-0.4791	0.1896			
APED(υ,α)	12.333	5.4576	192.5407	-381.0815	-374.7865
	12.4776	0.9920			
ED(υ)	7.6457	0.5829	177.8719	-353.7439	-350.5964

Table (4) presents the MLEs, log-likelihoods, AICs, and BICs of the ILWED and others. The Table indicate that ILWED is the best with minimum values of AIC and BIC. Furthermore, Table (5) indicated that the ILWED fitted the data well with small values of the Goodness-of-fit statistics.

Table 5: *The Goodness-of-fit statistics of the ILWED and others for the Italy Covid-19 Dataset*

Distributions	K-S	C-vM	A-D
ILWED	0.0593	0.1190	0.7196
WEED	0.0617	0.1526	0.9023
WExED	0.0738	0.2232	1.2740
TGED	0.1038	0.4544	2.4999
APED	0.0852	0.3133	1.7961
ED	0.1712	1.3955	7.0562

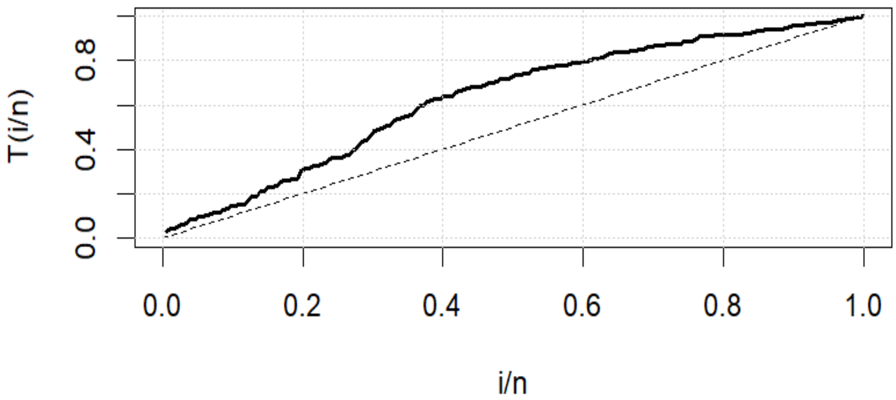

Figure 4: *The TTT-Plot for the Covid-19 Dataset*

The Total Time on Test (TTT) plot for the Covid-19 Dataset indicates an increasing hazard rate (concave shape), as seen in Figure (4).

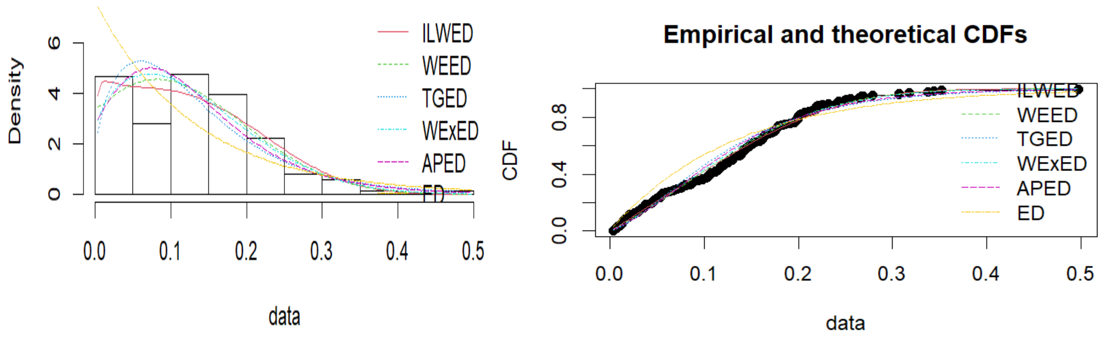

Figure 5 : Plots of the fitted PDFs a nd CDFs of the ILWED and other deistributions for the Covid-19 Dataset

Application to the **Airborne Communication Transceiver Dataset**

These data was reported by the [37], for the Repair Times in Hours for an Airborne Communication Transceiver. The dataset is as follows:

0.50, 0.60, 0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00, 1.00, 1.10, 1.30, 1.50, 1.50, 1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30, 4.00, 4.00, 4.50, 4.70, 5.00, 5.40, 5.40, 7.00, 7.50, 8.80, 9.00, 10.20, 22.00, 24.50.

Table 6: *MLEs and Goodness-of-fit Criteria for the fitted ILWED and other comparators for the comparators for the Airborne Dataset*

Distributions	Estimates	Standard Error	-ll	AIC	BIC
ILWED$(\upsilon,\alpha,\gamma,\lambda)$	1.1832	0.1923	89.2159	186.4318	193.1873
	0.0064	0.0040			
	5.6689	4.7566			
	0.0351	0.0175			
WILD$(\upsilon,\alpha,\gamma,\lambda)$	4.6910	1.7635	94.1066	194.2132	199.2798
	0.0078	0.0038			
	284.6967	211.7689			
	3.0914	3.3225			
TGED(υ,α,γ)	1.2234	0.2423	94.1066	194.2132	199.2798
	0.2086	0.0609			
	0.6441	0.2939			

$\text{APED}(\upsilon,\alpha)$	0.0156 0.0774	0.0833 0.1043	94.4479	192.8957	196.2735
$\text{WD}(\upsilon,\alpha)$	0.9604 3.9276	0.1089 0.6873	95.5114	195.0227	198.4005
$\text{ED}(\upsilon)$	0.2492	0.0394	95.5766	193.1532	194.8420

Table (6) presents the MLEs, log-likelihoods, AICs, and BICs of the ILWED and others. The Table indicate that ILWED is the best with minimum values of AIC and BIC. Furthermore, Table (7) indicated that the ILWED fitted the data well with small values of the Goodness-of-fit statistics.

Table 7*: The Goodness-of-fit statistics of the ILWED and others for the Airborne Dataset*

Distributions	K-S	C-vM	A-D
ILWED	0.0969	0.0597	0.3985
WILD	0.1230	0.0916	0.5908
TGED	0.1465	0.1300	0.9032
APED	0.1499	0.1003	1.0214
WD	0.1290	0.1359	1.0214
ED	0.1380	0.1574	1.0957

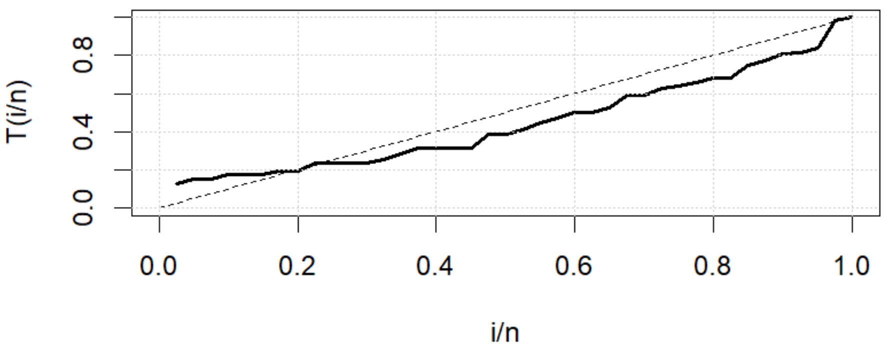

Figure 6: *The TTT-Plot for the Airborne Dataset*

The Total Time on Test (TTT) plot for the Airborne Dataset indicates a decreasing hazard rate (convex shape), as seen in Figure (6).

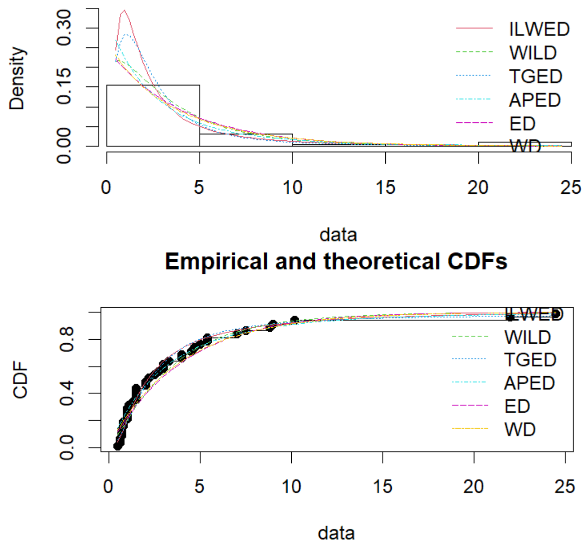

Figure 7: *The Fitted PDFs and CDFs of the ILWED and other distributions for the Airborne Dataset*

Figure (7)) indicates that the Strengths data is skewed to the right. ILWED fitted the data well compared with the other comparators.

Application to the Fatigue Fracture Dataset

These data was reported by [38], for the fatigue fracture of Kevlar 373/epoxy at fixed pressure until all had failed. The dataset is as follows:

0.0251, 0.0891, 0.0886, 0.3113, 0.8425, 0.2501, 0.4763, 0.5650, 0.5671, 0.3451, 0.8645, 0.6566, 0.6751, 0.6748, 0.8375, 0.7696, 0.6753, 0.8391, 0.9836, 0.8851, 0.9120, 0.9113, 1.0483, 1.0773, 1.1733, 1.0596, 1.5733, 1.7083, 1.2570, 1.7263, 1.2766, 1.2985, 1.7630, 1.3211, 1.3503, 1.9316, 1.3551, 1.4595, 1.8808, 1.4880, 1.5728, 1.8881, 1.7460, 1.8275, 1.8375, 1.7746, 1.8503, 1.8878, 1.9558, 2.2100, 2.0408, 2.0903, 2.1093, 2.1330, 2.2460, 2.0048, 2.2878, 2.3203, 2.3513, 2.4951, 2.3470, 2.9911, 2.5260, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.9143, 3.7455, 4.8073, 5.5295, 5.4005, 6.5541, 5.4435, 9.0960.

Table 8: *MLEs and Goodness-of-fit Criteria for the fitted ILWED and other comparators for the comparators for the Fatigue Fracture Dataset*

Distributions	Estimates	Standard Error	-ll	AIC	BIC
ILWED$(\upsilon,\alpha,\gamma,\lambda)$	2.6663	0.5229	120.4480	248.8953	258.2182
	0.0133	0.0189			
	0.5033	0.1606			
	0.0781	0.0301			
WILD$(\upsilon,\alpha,\gamma,\lambda)$	0.0184	0.0268	122.5050	253.0090	262.3319
	1.3408	0.1378			
	0.7831	0.8833			
	0.1398	0.2231			
IEOLED$(\upsilon,\alpha,\gamma,\lambda)$	0.1993	0.0352	127.6270	263.2534	272.5764
	1.0492	0.0719			
	549101.3	17416.71			
	54907.17	64.0385			
EOFWD$(\upsilon,\alpha,\gamma,\lambda)$	0.3409	0.2046	127.9080	263.8156	
	1.8342	0.4719			
	0.1399	0.1894			
	0.7365	0.2436			
WD(υ,θ)	1.3257	0.1138	122.5250	249.0494	253.7108
	2.1327	0.1945			
ED(υ)	0.5104	0.0585	127.1140	256.2287	258.5594

Table (8) presents the MLEs, log-likelihoods, AICs, and BICs of the ILWED and others. The Table indicate that ILWED is the best with minimum values of AIC and BIC. Furthermore, Table (9) indicated that the ILWED fitted the data well with small values of the Goodness-of-fit statistics.

Table 10: ***The Goodness-of-fit statistics of the ILOEED and others for the Fatigue Fracture Dataset***

Distributions	K-S	C-vM	A-D
ILWED	0.0747	0.0601	0.3849
WILD	0.1112	0.1363	0.7908
IEOLED	0.1436	0.4355	2.2674
WEED	0.0864	0.0859	0.5296
EOFWD	0.1297	0.3563	1.9729
WD	0.1099	0.1353	0.7887
ED	0.1663	0.57081	2.9881

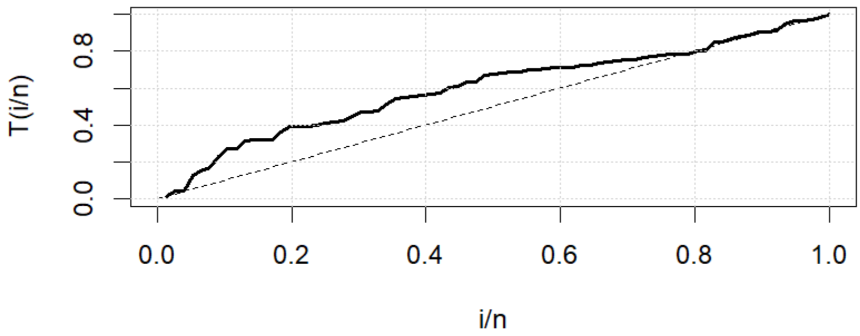

Figure 8: *The TTT-Plot for the Times to Fatigue Fracture Dataset*

The Total Time on Test (TTT) plot for the Fatigue Fracture Dataset indicates an increasing hazard rate, as seen in Figure (8).

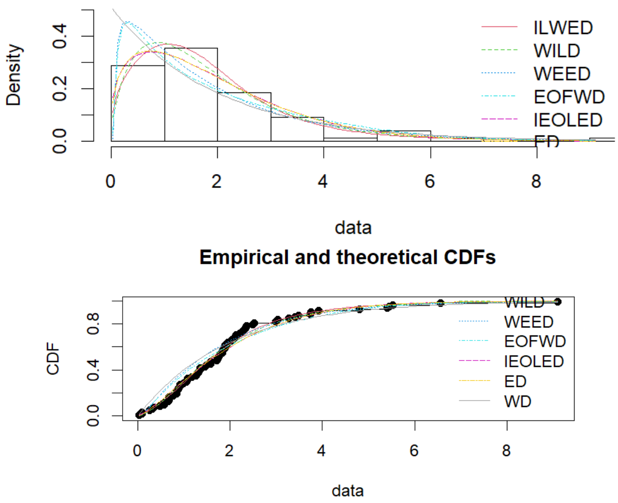

Figure 9: *The Fitted PDFs and CDFs of the ILWED and others for the Fatigue Fracture Dataset*

Figures (9) indicate that the Fatigue Fracture data is skewed to the right. ILWED fitted the data well compared with the other comparators.

Conclusion

In this research, we suggest and investigate a novel probability distribution that is a combination of the Inverse Lomax, Weibull, and exponential distributions, combining the properties of the three distributions. This merger is required if the data in issue combines both the Inverse Lomax, Weibull, and the exponential distributions' features described in

Section 1. We looked into some of its statistical properties, such as moments, the moment generating function, the characteristic function, and the quantile function. The parameters were determined using the maximum likelihood technique. According to the simulation studies, as the sample size grows, the estimations of the Biases and RMSEs approach zero, indicating that the estimates are more accurate. Three cases of parameter combination were considered for the simulation studies. The estimates were stable. Exemplifications of real-world datasets demonstrate the ILWED's significance. For the three datasets used, the proposed distribution is the best with minimum values of the Goodness-of-fit criteria and Goodness-of-fit statistics. This means ILWED can be used to fit datasets with increasing and decreasing hazard rates. Based on these facts, we hope that the ILWED will be preferred above the other models considered in this study. Only datasets from the industry were considered to fit the proposed distribution. We suggest that other areas should be explored in terms of the application of the proposed distribution. Also, other methods of estimation can be considered in further studies.

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