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A Theoretical Exploration of Paraletrix Calculus as an Extension of Rhotrix Mathematics

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Abstract

Building on earlier developments in generalized matrix theory, this paper advances the mathematical framework of *paraletrix* calculus as an extension of rhotrix mathematics. Previous studies introduced matrix-tertions, matrix-ngittrets, and thotrices as intermediary structures between conventional vector and matrix forms, while subsequent work in rhotrix theory established several multiplication techniques and related results. Recognizing the need for a more flexible structure capable of accommodating unequal numbers of rows and columns, this study focuses on the *paraletrix* as a generalization of the thotrix. The paper aims to extend this framework by introducing the concepts of differentiation and integration within *paraletrix* calculus and by defining these operations with respect to an independent variable in functional form. Through this theoretical exploration, the study contributes to the further development of generalized matrix theory by broadening the analytical scope of *paraletrix* structures and opening new possibilities for formal mathematical operations within this extended system.

Keywords: Paraletrix; Rhotrix Mathematics; Differentiation; Integration; Generalized Matrix Theory

Introduction

The concept of a thotrix a mathematical object whose components are systematically arranged in a rhomboidal or diamond-like structure was first introduced by Ajibade [2]. This innovation was presented as a natural extension of the pioneering ideas on matrix-tertions and matrix-noitrets developed by Atanassov and Shannon [1]. The introduction of thotrices represents a significant evolution in matrix theory, aiming to generalize and broaden the traditional properties and applications of matrices.

Rhotrices are mathematical entities that occupy an intermediate position between (2×2)- and (3×3)-dimensional matrices. Previous studies have produced various results on rhotrices, and further exploration of these structures led to the development of a new algebraic construct called the paraletrix, characterized by the arrangement of elements in a parallelogram-like form. This paper presents several findings related to the heart-oriented paraletrix ring, also referred to as the commutative paraletrix ring, extending known results from the rhotrix ring. Specifically, using the commutative (heart-oriented) rhotrix approach, we demonstrate that an integral calculus operation is valid for the heart-oriented paraletrix ring. Additionally, we establish analogues of Cayley's theorem and the arithmetic series within the framework of the heart-oriented paraletrix ring. The results obtained here further enrich the existing body of knowledge on paraletrix rings [7].

In classical linear algebra, matrices are well-known for being rectangular or square arrays of numbers, symbols, or expressions arranged in rows and columns. Their shape depends on the relationship between the number of rows and columns: a square matrix has an equal number of rows and columns, while a rectangular matrix does not. However, thotrices differ from conventional matrices in structure and conceptual design. A defining feature of a thotrix is that it always possesses an equal number of rows and columns, arranged in such a way that their overall formation assumes a rhomboidal configuration rather than the standard rectangular form of traditional matrices. This unique structure provides a new geometric and algebraic perspective that allows for a wider range of analytical and computational possibilities.

Ajibade [2] not only established the foundational definition of thotrices but also formalized the corresponding binary operations that govern their manipulation—specifically, addition (+) and multiplication (0). These operations, analogous to those defined for matrices, are carefully structured to maintain consistency with the rhomboidal nature of thotrices while ensuring algebraic closure within their defined set. To illustrate these operations, consider two thotrices, (R) and (Q), defined as follows: The paper would then proceed to formally express the mathematical definition and rules for the addition and multiplication of thotrices.

A rhotrix with an equal number of rows and columns is a well-studied concept in existing literature. The heart-oriented and row–column multiplication operations for such rhotrices are well established, and several results have been derived based on these operations. Inspired by the idea of both rectangular and square matrices, we propose a new structure called a paraletrix, which generalizes the rhotrix by allowing different numbers of rows and columns. We further explore its properties using heart-oriented and row–column multiplications. When the paraletrix has an equal number of rows and columns, we demonstrate that it coincides with the conventional rhotrix [6].

Through these definitions, Ajibade expanded the mathematical framework of matrix theory by introducing a new class of algebraic entities with potential applications in fields such as computational mathematics, data representation, and cryptographic modeling. The study of thotrices therefore contributes not only to theoretical mathematics but also to the practical exploration of higher-dimensional algebraic systems.

$$R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \text{ and } Q = \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle . \qquad (1)$$

Then the binary operation of addition (+) is given by:

$$R + Q = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & a+f & \\ b+g & h(R)+h(Q) & d+j \\ & e+k & \end{array} \right\rangle$$

and multiplication $(\circ)$ is

$$R \circ Q = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & ah(Q)+fh(R) & \\ bh(Q)+gh(R) & h(R)h(Q) & dh(Q)+jh(R) \\ & eh(Q)+kh(R) & \end{array} \right\rangle$$

Another multiplication method for rhotrices called *row-column multiplication* was introduced by Sani [3] to answer some questions raised by Ajibade [2]. The row-column multiplication method is in a similar way as that of multiplication of matrices and is illustrated using the rhotrices R and Q defined above in (1) as

$$R \circ Q = \left\langle \begin{array}{ccc} & af+dg & \\ bf+eg & h(R)h(Q) & aj+dk \\ & bj+ek & \end{array} \right\rangle.$$

A generalization of the row-column multiplication method for n-dimensional rhotrices was given by Sani [4]. That is, given n-dimensional rhotrices $R_n = \langle a_{ij}, c_{lk} \rangle$ and $Q_n = \langle b_{ij}, d_{lk} \rangle$ the multiplication of R_n and Q_n is as follows:

$$R_n \circ Q_n = \langle a_{i_1 j_1}, c_{l_1 k_1} \rangle \circ \langle b_{i_2 j_2}, d_{l_2 k_2} \rangle = \left\langle \sum_{i_2 j_1 = 1}^{t} (a_{i_1 j_1} b_{i_2 j_2}), \sum_{l_2 k_1 = 1}^{t-1} (c_{l_1 k_1} d_{l_2 k_2}) \right\rangle, \; t = (n+1)/2. \tag{2}$$

Rhotrix vectors (either row vectors or column vectors) can be represented in t different ways where $t = (n+1)/2$. This is different compared to vectors in matrices that can be represented in a unique way.

Matrices are either square or rectangular depending on the number of rows and columns while rhotrices always have the same number of rows and columns. Having this observation as a motivation, we introduced a structure called paraletrix as an object that has its elements arranged in a parallelogram whose number of rows and columns are not necessarily the same. To the best of our knowledge paraletrix has not received attention in the past and no work has been done on differentiation and integration of paraletrix. It is the main aim of this paper to present the concept of differentiation and integration of paraletrix, a generalization of rhotrix.

Materials and Method

Paraletrix and its basic Properties.

Let $m \text{ } and \text{ } n$ be the number of rows and columns of an arbitrary paraletrix where $m , n \in \{2k+1 : k \in \mathrm{N}\}$.

An $m \times n$ –dimensional paraletrix is of the form: $P_{m \times n} = \left\langle a_{ij,} c_{lk} \right\rangle$

$$= \left\langle \begin{array}{ccccccccc}
 & & & & a_{11} & & & & \\
 & & & a_{21} & c_{11} & a_{12} & & & \\
 & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
a_{\frac{m+1}{2}1} & c_{\frac{m-1}{2}1} & \cdots & \cdots & h(p) & \cdots & \cdots & c_{1\frac{n-1}{2}} a_{1\frac{n+1}{2}} & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & a_{\frac{m+1}{2}\frac{n-3}{2}} & c_{\frac{m-1}{2}\frac{n-3}{2}} & a_{\frac{m-1}{2}\frac{n-1}{2}} & c_{\frac{m-3}{2}\frac{n-1}{2}} & a_{\frac{m-3}{2}\frac{n+1}{2}} & & \\
 & & & a_{\frac{m+1}{2}\frac{n-1}{2}} & c_{\frac{m-1}{2}\frac{n-1}{2}} & a_{\frac{m-1}{2}\frac{n+1}{2}} & & & \\
 & & & & a_{\frac{m+1}{2}\frac{n+1}{2}} & & & &
\end{array} \right\rangle \quad (3)$$

Where $a_{ij}, c_{ik} \in R$ for

$$i = 1,2,\ldots,\frac{m+1}{2}, j = 1,2,\ldots,\frac{n+1}{2} \; and \quad i = 1,2,\ldots,\frac{m-1}{2}, k = 1,2,\ldots,\frac{n-1}{2}$$

For example,

$$P_{3\times5} = \left\langle \begin{array}{cccc}
 & a_{11} & & \\
a_{21} & c_{11} & a_{12} & \\
 & a_{22} & c_{12} & a_{13} \\
 & & a_{23} &
\end{array} \right\rangle \quad (4)$$

Equation (4) is a paraletrix with three rows:

$$\begin{array}{ccccccc}
a_{11} & & c_{11} & \mathit{and} & a_{21} & & \\
& a_{12} & & c_{12} & & a_{22} & \\
& & a_{13} & & & & a_{23}
\end{array}$$

and five columns:

$$\begin{array}{cccccccc}
& a_{11} & & & a_{12} & & \mathit{and} & a_{13} \\
a_{21} & & c_{11} & a_{22} & & c_{12} & a_{23} &
\end{array}$$

It is important to mention here that the name paraletrix is as a result of the parallelogram nature of the arrangement of its entries. From the paraletrix defined in (3), we can extract two matrices:

$A = (a_{ij})$ and $C = (c_{lj})$ where $i = 1, 2, . . . , \frac{m+1}{2}$, $j = 1, 2, . . . , \frac{n+1}{2}$, and $k = 1, 2, . . . , \frac{m-1}{2}$, $l = 1, 2, . . . , \frac{n-1}{2}$. Matrix $A = (a_{ij})$ is called the major matrix and $C = (c_{lj})$ is the minor matrix and are either square or rectangular depending on the dimension of the paraletrix.

The Heart of a Paraletrix

The Heart (or centre) of a paraletrix is simply defined as the element in a paraletrix which divides the entries into two equal parts. In other words, the heart of an $m \times n$ -dimensional paraletrix $P_{m\times n}$ is the element located at the $\frac{1}{2}[\frac{1}{2}((m \times n) + 1) + 1]$ position of the paraletrix. It worthy to mention here that not every paraletrix has a heart, A. Aminu and O. Michael [5], and this is addressed in the following lemma.

Lemma 1.

Let $P_{m\times n}$ be an $m \times n$ –dimensional paraletrix. If $P_{m\times n}$ has a heart then the heart is unique.

Proof. By contradiction, suppose h_1 and h_2 are hearts of $P_{m\times n}$ such that $h_1 \neq h_2$. It follows from the definition of heart that h_1 and h_2 lies in the $\frac{1}{2}[\frac{1}{2}((m \times n) + 1) + 1]$ position of $P_{m\times n}$. Since the position is unique then $h_1 = h_2$ and the statement follows.

Paraletrix Differentiation

If the elements of a paraletrix $P_{m\times n}$ are functions of a variable x, then the paraletrix is called a paraletrix function of x; that is:

$P_{m\times n} = P_{m\times n}(x) = \langle a_{ij}(x), c_{lk}(x)\rangle$ and the differential coefficients of $P_{m\times n}$ with respect to x is

$$\frac{d}{dx}P_{m\times n}(x) = \left\langle \frac{d}{dx}a_{ij}(x), \frac{d}{dx}c_{lk}(x)\right\rangle$$

And the nth order derivative is given as:

$$\frac{d^n}{dx^n}P_{m\times n}(x) = \left\langle \frac{d^n}{dx^n}a_{ij}(x), \frac{d^n}{dx^n}c_{lk}(x)\right\rangle, \qquad for \qquad n = 1,2,3,\ldots$$

Thus the elements of the differentiated paraletrix $\frac{d}{dx}P_{m\times n}(x)$ are the derivatives of the corresponding elements of $P_{m\times n}(x)$ of equation (3) .

Theorem 1. (Derivative of sum of two paraletrix)

Let $P_{m\times n}(x)$ *and* $Q_{r\times s}(x)$ be two paraletrices, each with differentiable elements. Then

$$\frac{d}{dx}[P_{m\times n}(x) + Q_{r\times s}(x)] = \left\langle \frac{d}{dx}P_{m\times n}(x) + \frac{d}{dx}Q_{r\times s}(x)\right\rangle$$

Proof. The proof is obvious from the definition of sum of two paraletrices.

Theorem 2. (Derivative of the product of two paraletrix)

Let $P_{m\times n}(x)$ *and* $Q_{r\times s}(x)$ be two paraletrices which are compatible for multiplication, each with differentiable elements. Then

$$\frac{d}{dx}[P_{m\times n}(x)Q_{r\times s}(x)] = \left\langle P_{m\times n}(x)\frac{d}{dx}Q_{r\times s}(x) + Q_{r\times s}(x)\frac{d}{dx}P_{m\times n}(x)\right\rangle$$

Proof. The proof follows immediately from the definition of the paraletrix multiplication.

Example 1. Find $\frac{d}{dx}P_{3\times 5}(x)$**,** given that

$$P_{3\times 5}(x) = \left\langle \begin{matrix} & -x^3 & & \\ \frac{1}{x} & 1-x & 7x & \\ & \cos x & -5 & e^{3x} \\ & & \sin x & \end{matrix} \right\rangle$$

Solution

$$\frac{d}{dx}P_{3\times 5}(x)=\left\langle \begin{array}{llll} & \frac{d}{dx}(-x^3) & & \\ \frac{d}{dx}\frac{1}{x} & \frac{d}{dx}(1-x) & \frac{d}{dx}7x & \\ & \frac{d}{dx}\cos x & \frac{d}{dx}(-5) & \frac{d}{dx}(e^{3x}) \\ & & \frac{d}{dx}\sin x & \end{array} \right\rangle$$

$$=\left\langle \begin{array}{llll} & -3x^2 & & \\ -\frac{1}{x^2} & -1 & 7 & \\ & -\sin x & 0 & 3e^{3x} \\ & & \cos x & \end{array} \right\rangle$$

Paraletrix Integration

Just as we can find the derivative of a paraletrix $P_{m\times n}$ with respect to an independent variable x, we can as well find the anti-derivative or the definite integral of $P_{m\times n}$. The integral of $P_{m\times n}(x)$, either definite or indefinite, is obtained by integrating each element of $P_{m\times n}(x)$ as follows:

$$\int_a^b P_{m\times n}(x)dx=\left\langle [\int_a^b a_{ij}(x),c_{lk}(x)]dx \right\rangle=\left\langle \int_a^b a_{ij}(x)dx,\int_a^b c_{lk}(x)dx \right\rangle \text{ and}$$

$$\int P_{m\times n}(x)dx=\left\langle [\int a_{ij}(x),c_{lk}(x)]dx \right\rangle=\left\langle \int a_{ij}(x)dx,\int c_{lk}(x)dx \right\rangle$$

Thus we equally integrate equation (3) just as we did in differentiation.

Theorem 4.

Let $P_{m\times n}(x)$ *and* $Q_{r\times s}(x)$ be two paraletrices. Then

$$\int P_{m\times n}(x)dx+\int Q_{r\times s}(x)dx=\left\langle \int P_{m\times n}(x)dx+\int Q_{r\times s}(x)dx \right\rangle$$

Proof. This follows immediately from the definition of sum of two paraletrices.

Example 3.

Find $\int P_{3\times5}(x)$; where $\int P_{3\times5}(x)$ is as defined above in example 1.

Solution.

$$\int P_{3\times5}(x) = \int [\left\langle \begin{array}{cccc} & -x^3 & & \\ \frac{1}{x} & 1-x & 7x & \\ & \cos x & -5 & e^{3x} \\ & & \sin x & \end{array} \right\rangle] dx$$

$$= \left\langle \begin{array}{cccc} & \int(-x^3)dx & & \\ \int \frac{1}{x}dx & \int(1-x)dx & \int 7xdx & \\ & \int \cos xdx & \int(-5)dx & \int e^{3x}dx \\ & & \int \sin xdx & \end{array} \right\rangle$$

$$\left\langle \begin{array}{cccc} & -\frac{x^4}{4}+C_1 & & \\ \ln x + C_6 & x-\frac{x^2}{2}+C_4 & \frac{7x^2}{2}+C_2 & \\ & \sin x + C_7 & -5x+C_5 & \frac{e^{3x}}{3}+C_3 \\ & & -\cos x + C_8 & \end{array} \right\rangle$$

Conclusion

We have presented the concept of differentiation and integration of paraletrix, a generalization of rhotrix. It is worthy to mention that a paraletrix can only be added or multiplied together if they have the same dimension and also if the heart exist. To the best of our knowledge, paraletrix has not received attention in the past and the concept of differentiation and integration of paraletrix has not surface elsewhere in recent times. This paper is open to further investigation.

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