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Quantum Wreath Products of Transformation Semigroups: Associativity, Distributivity, and Faithful Embeddings into C*-Algebras

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Abstract

Connecting discrete semigroup theory with quantum C*-algebras requires a rigorous account of how semigroup constructions are preserved in operator-algebraic settings. This study develops a framework for embedding the direct and wreath products of transformation semigroups into quantum C*-algebras. Through algebraic verification and operator-theoretic construction, the study first establishes that the wreath product of transformation semigroups is associative and distributes over the direct product under appropriate conditions. Faithful representations of these products are then constructed as operators on Hilbert spaces, yielding explicit C*-algebraic isomorphisms. The results show that the direct product admits an embedding as a direct sum of C*-algebras, whereas the wreath product corresponds to a crossed product involving tensor products and the C*-algebras of the acting semigroup. They further demonstrate that semigroup-level isomorphisms lift to the associated C*-algebras and that the wreath product distributes over direct sums in the C*-algebraic setting. Necessary and sufficient conditions are established under which these embeddings preserve both algebraic structures and operator-theoretic properties. By connecting transformation semigroup constructions with noncommutative geometry, this framework provides an operator-algebraic foundation for analyzing direct and wreath products through faithful quantum representations.

Keywords: C*-Algebras; Crossed Products; Direct Products; Transformation Semigroups; Wreath Products

INTRODUCTION

The theory of transformation semigroups occupies a fundamental position in algebraic structure theory, with applications ranging from automata theory to topological dynamics. While the individual definitions and basic properties of the wreath product (denoted by $\wr$ or Wrt) and direct product $(\times)$ for transformation semigroups are well-established in the classical literature (Clifford & Preston, 1961; Meldrum, 1995), a rigorous investigation into their fundamental interactive properties specifically associativity of the wreath product and its distributivity over the direct product has remained an area requiring further development within a modern mathematical framework.

Kumar (2012) initiated an important line of inquiry by proving elementary results concerning these properties, while Majumdar, Dey, and Hossain (2012) extended this work to certain classes of semigroups. However, as noted in the proposal, these results necessitate thorough verification and integration into contemporary mathematics, particularly in light of recent advances in operator algebras and non-commutative geometry.

Concurrently, a significant parallel development has emerged at the interface of discrete algebraic structures and analytic operator algebras. Recent literature demonstrates growing interest in representing semigroups and their products within quantum mechanics and non-commutative geometry, specifically by embedding them into C^*-algebras (Kaliszewski, Quigg, & Robertson, 2022; Liu, 2024; Yashin, 2022). The work of Shevlyakov (2024) on wreath products and Plotkin's problem, together with Maksymenko's (2024) investigation of topological actions of wreath products, underscores the contemporary relevance of these constructions. Yet, a clear pathway from purely algebraic properties of semigroup products to their faithful representation as operators on Hilbert space has not been fully articulated.

This paper addresses this gap by developing a robust embedding mechanism that translates semigroup-theoretic constructions into the realm of quantum C^*-algebras. We construct faithful representations and determine necessary and sufficient conditions such as finiteness of the semigroups, properties of their state spaces, and preservation of semigroup structure under operator composition and adjoint for such embeddings to exist. Our approach builds upon the foundational categorical perspective of Rhodes and Steinberg (2009) while incorporating recent advances in semigroup C-algebras (Li, 2021).

The originality of this work lies in its direct focus on associativity and distributivity in the context of embedding into C^*-algebras, revealing how classical semigroup-theoretic isomorphisms lift to *-isomorphisms of operator algebras. This integration not only verifies and extends Kumar's (2012) results but also places them within the broader framework of non-commutative geometry, establishing necessary conditions for faithful representability.

MATERIALS AND METHODS

Review of Foundational Literature

Recent literature demonstrates a resurgence of interest in the categorical and topological properties of semigroup products. Shevlyakov (2024) has extended classical results on semigroup wreath products to address Plotkin's problem concerning algebraic structures over groups. Maksymenko (2024) has investigated topological actions of wreath products, and have enhanced our understanding of their dynamical representations and providing new insights into their continuous analogues.

On the analytic side, Liu (2024) studied representations of semigroups in operator systems, particularly examining completely positive extensions into C-algebras, while Yashin (2022) investigated boundary representations and the C-envelope of operator systems arising from semigroups. The historical foundation provided by Clifford and Preston (1961) and the modern categorical treatment by Rhodes and Steinberg (2009) continue to influence contemporary semigroup theory.

The work of Kaliszewski, Quigg, and Robertson (2022) on co-universal C-algebras associated with product systems has particular relevance to our embedding construction, as it provides a framework for understanding how semigroup actions lift to crossed product constructions. Similarly, Li's (2021) development of K-theory for semigroup C-algebras and partial crossed products offers essential tools for analyzing the structure of the algebras generated by our embeddings.

Definitions and Preliminary Results

Definition 1 (Direct Product). The direct product of a semigroups $S_1$and S_2 given as $S_1 \times S_2$, is defined as a transformation semigroup on $X_1 \times X_2$, the elements of $S_1 \times S_2$being the ordered pairs (s_1, s_2), $s_1 \in S_1$, $s_2 \in S_2$ with $(s_1, s_2)(x_1) = s_1(x_1)$, $(s_1, s_2)(x_2) = s_2(x_2)$, for each $x_1 \in X_1$, $x_2 \in X_2$. The multiplication in $S_1 \times S_2$ is component-wise. It will easily be seen that $S_1 \times S_2$ is indeed a transformation semigroup. If $S_1$and S_2 are finite, the number of elements of $S_1 \times S_2$ is obviously the product of the numbers of elements of $S_1 and\ S_2$

Definition 2 (Wreath Product). Let (S_1, X_1) and (S_2, X_2) be transformation semigroups. Their (unrestricted) wreath product, denoted $(S_1 \varsigma S_2, X_1 \times X_2)$, is defined as follows. The base set is the Cartesian product $X_1 \times X_2$. The semigroup $S_1 \varsigma S_2$ consists of all pairs (f, s_2) where $s_2 \in S_2$ and $f: X_2 \to S_1$ is a function. The action is given by:

$(f, s_2)(x_1, x_2) = (f(x_2)(x_1), s_2(x_2))$.

Multiplication is defined by $(f, s_2)(g, t_2) = (f._{s_2} g, s_2 t_2)$, where $(f._{s_2} g)(x_2) = f(x_2) g(s_2^{-1}(x_2)$ when inverses exist, and appropriate adjustments are made for non-invertible transformations.

Proposition (Kumar, 2012). The transformation wreath product of semigroups is associative.

Example: If $(S_1, X_1)$and $(S_2, X_2)$are transformation semigroups with $S_2 = \{1_{x_2}\}$, then $(S_1 \times S_2, X_1 \cup X_2)$ and $(S_1 \varsigma S_2, X_1 \times X_2)$ may be identified with $(S_1, X_1)$and $\left(\left(\prod_{x_2 \in X_2} S_{1,x_2}\right), \cup_{x_2 \in X_2} X_{1,x_2}\right)$ respectively, ignoring the trivial action of S_2 on X_2. Here, each $S_{1,x_2} \cong S_1$and each is in 1-1 correspondence with X_1 with $x_{1,x_2} \leftrightarrow x_1$ and $s_{1,x_2}(x_{1,x_2}) = s_1(x_1)$. Thus, in this case, $S_1 \times S_2 \cong S_1$and $S_1 \varsigma S_2 \cong \left(\prod_{x_2 \in X_2} S_{1,x_2}\right)$(direct product) as semigroups.

Proposition (Kumar, 2012**):** If (S_1, X_1), (S_2, X_2), (S_3, X_3) are three transformation semigroups, then $((S_1 \varsigma S_2) \varsigma S_3, (X_1 \times X_2) \times X_3) \cong (S_1 \varsigma (S_2 \varsigma S_3), X_1 \times (X_2 \times X_3)$.

Definition 3 (Li 2021): A Banach *-algebra is a Banach algebra A (a complete normed algebra) with an involution $* : A \rightarrow A$ that satisfies for all a, b in A and $\lambda \; in \; \mathbb{C}$:

1. $(a *) * = a$ (Involution)
2. $(a + b) * = a * + b *$
3. $(\lambda a) * = \{\bar{\lambda}\} \, a *$ (Conjugate linear)
4. $(ab) * = b * \, a *$

A $C^* - algebra$ is a Banach $* - algebra$ that additionally satisfies the following identity, which links the norm, the algebraic structure, and the involution:$C^* - Identity$: $||a * a|| = ||a||^2$ for all a in A. This identity is crucial and implies that the norm is uniquely determined by the algebraic structure. A fundamental example is the algebra $B(H)$ of all bounded operators on a Hilbert space H, where the involution $*$ is the Hilbert space adjoint and the norm is the operator norm.

RESULTS

Embedding Into C^*-Algebras

The Direct Product Embedding

For a transformation semigroup (S, X), we consider its regular representation on the Hilbert space $\ell^2(X)$). Each $s \in S$ defines a bounded linear operator

$$T_s \rightarrow \mathcal{B}\left(\ell^2(X)\right)$$

$$\text{by } T_s \, \xi(x) = \xi(s(x)) \text{ for } \xi \in \ell^2(X),$$

with appropriate modifications for non-bijective transformations. The C^*-algebra generated by $\{T_s : s \in S\}$ is denoted $C^*(S)$.

Theorem 1 (Direct Product Embedding). Let (S_1, X_1) and (S_2, X_2), be finite transformation semigroups. Then the direct product $S_1 \times S_2$ embeds faithfully into $B\left(\ell^2(X_1 \oplus X_2)\right)$ via the map:

$$\pi(s_1, s_2) = T_{s_1} \oplus T_{s_2} \in \mathcal{B}\left(\ell^2(X_1) \oplus \ell^2(X_2)\right)$$

The C^*-algebra generated by $\{ \pi(s_1, s_2) : (s_1, s_2) \in (S_1 \times S_2) \}$ is *-isomorphic to $C^*(S_1) \oplus C^*(S_2)$.

Proof: The map π is clearly injective and preserves multiplication since $(T_{s_1} \oplus T_{s_2})(T_{t_1} \oplus T_{t_2})= (T_{s_1t_1} \oplus T_{s_2t_2})$. The generated C^*-algebra consists of operators of the form $A \oplus B$ where $A \in C^*(S_1)$ and $B \in C^*(S_2)$), giving the stated isomorphism.

The Wreath Product Embedding

The wreath product requires a more sophisticated construction, building on the work of Kaliszewski et al. (2022) on product systems and crossed products.

Construction 1: (Wreath Product Embedding). Let (S_1, X_1) and (S_2, X_2) be finite transformation semigroups with $|X_2| = n$. Denote $A \in C^*(S_1)$. Consider the n-fold tensor product:

$$A^n = \otimes_{x_1 \in X_1} A_{x_2},$$

where each $A_{x_2} \cong A$. The semigroup S_2 acts on X_2 by permutations (assuming faithful action), inducing an action $\alpha: S_2 \to Aut(A^n)$ by permuting tensor factors: $\alpha_{s_2}(a_{x_2}) = a_{s_2(x_2)}$

The wreath product $S_1 \wr S_2$ embeds into the crossed product $A^n \rtimes_\alpha C^*(S_2)$ via the map:

$$\Phi(f, s_2) = \left(\otimes_{x_1 \in X_1} T_{f(x_2)}\right) \cdot \lambda_{s_2}$$

where λ_{s_2} is the implementing unitary for the action in the crossed product.

Theorem 2 (Wreath Product C^*-Algebras). The map Φ is a faithful representation of $S_1 \wr S_2$ into $A^n \rtimes_\alpha C^*(S_2)$. The C-algebra generated by $\Phi(S_1 \wr S_2)$ is *-isomorphic to into $A^n \rtimes_\alpha C^*(S_2)$.

Proof: The proof proceeds by verifying that \Phi preserves the wreath product multiplication:

$$\Phi(f, s_2)\Phi(g, t_2) = \left(\otimes_{x_2} T_{f(x_2)}\right) \cdot \lambda_{s_2} \left(\otimes_{x_2} T_{g(x_2)}\right) \cdot \lambda_{t_2}$$

Using the covariance relation$\lambda_{s_2} \left(\otimes_{x_2} T_{g(x_2)}\right) \lambda^*_{s_2} = \otimes_{x_2} T_{g(s_2^{-1}(x_2))}$, we obtain:

$$\left(\otimes_{x_2} T_{f(x_2)}\right)\left(\otimes_{x_2} T_{g(s_2^{-1}(x_2))}\right)\lambda_{s_2t_2} = \left(\otimes_{x_2} T_{f(x_2)g(s_2^{-1}(x_2))}\right)\lambda_{s_2t_2} = \Phi(f \cdot_{s_2} g,\ s_2t_2)$$

Injectivity follows from the faithfulness of the regular representation of $S_1$and the crossed product construction.

Associativity in the C^*-Algebras Context

The classical associativity of the wreath product lifts to a -isomorphism of the corresponding C-algebras.

Theorem 3 (C^*-algebras Associativity of Wreath Product). Let (S_1, X_1), (S_2, X_2), and (S_3, X_3) be finite transformation semigroups with $|X_2| = m$ and $|X_3| = n$. Let $A = C^*(S_1)$. Then there is a natural *-isomorphism:

$$((A^m)^n \rtimes C^*(S_2)) \rtimes C^*(S_3) \cong A^{(m.n)} \rtimes (C^*(S_2) \rtimes C^*(S_3)),$$

where the actions are induced by the permutation actions on the index sets. Consequently, the C*-algebra generated by $(S_1 \wr S_2) \wr S_3$ is *-isomorphic to that generated by $S_1 \wr (S^2 \wr S_3)$

Proof. The left-hand side corresponds to $(S_1 \wr S_2) \wr S_3$ via successive applications of Theorem 2. The iterated crossed product

$(A^m \rtimes C^*(S_2))^n \rtimes C^*(S_3)$ is isomorphic to

$$A^{mn} \rtimes (C^*(S_2) \rtimes C^*(S_3))$$

by the associativity of the crossed product construction for amenable semigroups (Li, 2021). The right-hand side corresponds to $S_1 \wr (S^2 \wr S_3)$), completing the proof.

Distributivity Properties

We now establish how the wreath product distributes over direct sums in the C*-algebraic context.

Theorem 4 (Distributivity over Direct Sums). Let (S_1, X_1) be a transformation semigroup and let (S_2, X_2), (S_3, X_3) be transformation semigroups with disjoint action sets. Then there is a natural isomorphism of transformation semigroups:

$$S_1 \wr (S_2 \times S_3) \cong (S_1 \wr S_2) \times (S_1 \wr S_3)$$

where the direct product on the right-hand side is taken in the category of transformation semigroups. This lifts to a -isomorphism of C-algebras:

$$C^*(S_1)^{|X_2|+|X_3|} \rtimes C^*(S_2 \times S_3) \cong C^*(S_1)^{|X_2|} \rtimes C^*(S_2) \oplus (C^*(S_1)^{|X_3|} \rtimes C^*(S_3))$$

Proof: The algebraic isomorphism follows from the definition of the wreath product and the fact that functions from $X_2 \times X_3$ to S_1 decompose as pairs of functions from X_2 and X_3 respectively. The C^*-algebraic isomorphism follows from the corresponding decomposition of tensor products and crossed products.

Necessary and Sufficient Conditions for Faithful Embedding

The existence of faithful embeddings requires certain conditions on the semigroups involved.

Theorem 5 (Embedding Conditions): Let (S, X) be a transformation semigroup. Then S admits a faithful embedding into $B\left(\ell^2(X)\right)$ as described above if and only if:

1. S is finite, or more generally, has a faithful finite-dimensional representation;
2. The action of S on X is effective (i.e., different elements induce different transformations);
3. The state space of S admits sufficiently many pure states to separate points.

For the wreath product $S_1 \wr S_2$ to embed faithfully, it suffices that both S_1 and S_2 satisfy these conditions and that the action of S_2 on X_2 is by permutations (or more generally, by partial bijections that admit a C^*-algebraic implementation).

Proof: Condition (1) ensures that the regular representation yields bounded operators; (2) guarantees injectivity of the representation; (3) ensures that the generated C^*-algebra separates points of S. The wreath product condition follows from Theorem 2 and the fact that the crossed product construction preserves faithfulness.

DISCUSSION

The results presented in Section 3 establish a rigorous bridge between classical semigroup theory and modern operator algebra. Theorem 2 confirms that Kumar's (2012) algebraic associativity lifts to the C*-algebraic setting, while Theorem 3 provides a new distributivity result that has no direct analogue in the purely algebraic theory. These findings have several important implications.

First, they validate the approach of representing semigroup products through crossed products of tensor products, showing that this construction preserves the essential structural properties. This addresses the gap identified in the introduction concerning the need for a clear pathway from algebra to analysis.

Second, the necessary and sufficient conditions in Theorem 4 clarify the limitations of such embeddings. Not all transformation semigroups admit faithful C*-algebraic representations; finiteness (or more generally, the existence of faithful finite-dimensional representations) is essential. This explains why previous work (Kaliszewski et al., 2022; Liu, 2024) has focused on finite or amenable cases.

Third, the distributivity result (Theorem 4) reveals a deeper categorical structure: the wreath product behaves like a tensor product in a monoidal category, with the direct sum playing the role of coproduct. This suggests potential connections to the work of Rhodes and Steinberg (2009) on the q-theory of finite semigroups and its categorical foundations.

The work of Shevlyakov (2024) on wreath products and Plotkin's problem takes on new significance in light of these results. The C*-algebraic representations constructed here provide concrete realizations of the algebraic structures studied in that work, potentially enabling analytic methods to be applied to problems in universal algebra.

Similarly, Maksymenko's (2024) investigation of topological actions of wreath products finds a natural home in the crossed product construction, where continuous actions on C*-algebras generalize discrete permutation actions. The embedding theorem (2) can be extended to topological semigroups acting on topological spaces, yielding continuous analogues of the discrete construction.

CONCLUSION

This paper has established a comprehensive framework for embedding transformation semigroup products into C^*-algebras. We have verified and extended Kumar's (2012) results on associativity of the wreath product, proving that these algebraic isomorphisms lift to -isomorphisms of the corresponding operator algebras. The direct product embeds as a direct sum of C^*-algebras, while the wreath product corresponds to a crossed product of tensor products with the acting semigroup's C^*-algebras.

Our main contributions are:

1. A faithful representation of the direct product $S_1 \times S_2$ as $(C^*(S_1)\oplus(C^*(S_2)$ (Theorem 1);

2. A faithful representation of the wreath product $S_1 \wr S_2$ as the crossed product $C^*(S_1)^{|X_2|} \rtimes C^*(S_2)$ (Theorem 2);
3. Proof that associativity $(S_1 \wr S_2) \wr S_3 \cong S_1 \wr (S_2 \wr S_3)$ to a -isomorphism of C-algebras (Theorem 3);
4. Establishment of a distributivity law $S_1 \wr (S_2 \times S_3) \cong (S_1 \wr S_2) \times (S_1 \wr S_3)$ and its C^*-algebraic analogue (Theorem 4);
5. Identification of necessary and sufficient conditions for faithful embeddability (Theorem 5).

These results bridge discrete semigroup theory with non-commutative geometry, providing a concrete realization of wreath products as operator algebras and opening new avenues for applying analytic methods to semigroup-theoretic problems.

Recommendations

Based on the findings of this research, the following directions for future investigation are recommended:

1. Extension to infinite semigroups: Develop analogous embedding theorems for infinite transformation semigroups with appropriate topological or measure-theoretic structures, building on the work of Li (2021) on semigroup C^*-algebras and partial crossed products.
2. K-theoretic analysis: Compute the K-theory of the C^*-algebras arising from wreath product embeddings, following the framework established by Li (2021) and relating these computations to semigroup invariants.
3. Continuous analogues: Generalize the construction to topological semigroups acting on topological spaces, yielding continuous wreath products and their C*-algebraic counterparts, extending Maksymenko's (2024) work on topological actions.
4. Applications to Plotkin's problem: Apply the C^*-algebraic representations developed here to address open questions in universal algebra, particularly those concerning the structure of wreath products in equational theories, as studied by Shevlyakov (2024).
5. Quantum probability interpretations: Investigate the interpretation of wreath product C^*-algebras in terms of quantum stochastic processes, where the tensor product structure corresponds to independence and the crossed product to Markovian dynamics.

6. Computational implementations: Develop algorithmic methods for constructing explicit matrix representations of finite wreath products, enabling computational exploration of their properties and applications in quantum information theory.

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