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A Modified New Iterative Method for Solving Nonlinear Fractional-Order Delay Differential Equations

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Abstract

This paper explores the application of the Modified New Iterative Method (MNIM) to solve nonlinear fractional-order delay differential equations (NFDDEs). A series of test problems are presented to evaluate the method's performance across various fractional orders. The results indicate that MNIM yields highly accurate approximations, particularly when the fractional order approaches an integer. The method is especially effective for integer-order cases and for fractional orders close to them. However, its accuracy decreases as the fractional order becomes smaller, with noticeable errors emerging over larger domains. MNIM remains a powerful and adaptable approach for solving a broad class of fractional differential equations.

Keywords: Modified New Iterative Method (MNIM); Fractional orders

INTRODUCTION

This section provides a comprehensive overview of existing approaches for solving fractional delay differential equations (FDDEs), highlights the motivation for this study, identifies gaps in the current literature, and outlines the aim, objectives, scope, and limitations of the research.

Fractional delay differential equations (FDDEs) incorporate both fractional derivatives and time-delay components. Unlike their integer-order counterparts, fractional derivatives exhibit non-local behavior, capturing memory and hereditary effects. Time-delay terms, on the other hand, account for historical system states, thereby enhancing the modeling of dynamic processes. The combined use of these features makes FDDEs particularly suitable for describing complex, real-world systems with high fidelity. Consequently, FDDEs have found wide-ranging applications across disciplines such as physics, chemistry, control theory, electrochemistry, bioengineering, and population dynamics (Jhinga & Daftardar-Gejji, 2019; Srivastava, 2020; Du, 2022; El-Kalla et al., 2019; Avcı, 2022). In bioengineering, for example, fractional models provide deeper insights into the dynamic behavior of biological tissues and improve the interpretation of imaging techniques like nuclear magnetic resonance (NMR) and magnetic resonance imaging (MRI).

Despite their modeling advantages, FDDEs pose significant computational challenges due to the non-local nature of fractional derivatives. This has spurred ongoing research into the development of efficient, accurate, and time-saving numerical methods for solving nonlinear FDDEs. Among early contributions, Diethelm et al. (2002, 2004) extended the classical Adams-Bashforth scheme to develop the fractional Adams method (FAM), which was later adapted by Bhalekar and Daftardar-Gejji (2011a) to accommodate delay terms. Similarly, the Numerical Predictor-Corrector Method (NPCM), grounded in the Daftardar-Gejji and Jafari (DGJ) framework (2006), was introduced to improve computational efficiency in solving FDEs (Daftardar-Gejji et al., 2014), and subsequently extended to FDDEs (Daftardar-Gejji et al., 2015).

More recently, Kumar and Methi (2021) introduced a hybrid approach known as the Banach Contraction Method (BCM), which integrates the Banach fixed-point theorem with the New Iterative Method (NIM) originally proposed by Daftardar-Gejji and Jafari (2006). This hybrid technique has demonstrated superior accuracy and faster convergence compared to traditional methods such as FAM and NPCM.

Building on these developments, the present study introduces a Modified New Iterative Method (MNIM) tailored for solving FDDEs. While fractional operators inherently increase computational complexity, they remain essential for accurately capturing memory-driven dynamics in natural systems. The MNIM is designed to address the limitations of existing techniques including NIM, the Variational Iteration Method (VIM), FAM, and NPCM—by providing faster convergence, reduced computational overhead, and improved solution accuracy. This method aspires to serve as a more effective numerical tool for solving both linear and nonlinear FDDEs with enhanced precision and efficiency.

METHODOLOGY

Exploring the Fundamentals of the New Iterative Method (NIM)

In order to clarify the underlying principles of the initial approach in the New Iterative Method (NIM), one can draw insights from a well-established functional equation found in the works of Daftardar-gejji & Bhalekar (2010), Ramadan & Al-luhaibi (2015), Moltot & Deresse, (2022) and Ashitha & Ranjini (2020). This perspective begins with the examination of the non-linear functional equation introduced by Daftardar-gejji & Jafari, (2006).

$$y(x) = g(x) + N[y(x)] \qquad ...(1)$$

In this context, N represents the non-linear operator, and g is a known function. Our objective is to find a solution, denoted as $y(x)$, which possesses a series representation in the following format:

$$y = \sum_{i=0}^{\infty} y_i \,. \qquad ...(2)$$

The nonlinear operator N can be decomposed as

$$N\left(\sum_{i=0}^{\infty} y_i\right) = N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\}. \qquad ...(3)$$

From Eqns. (2) and (3), Eqn. (1) is equivalent to

$$\sum_{i=0}^{\infty} y_i = g + N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_j\right) - N\left(\sum_{j=0}^{i-1} y_j\right)\right\}. \qquad ...(4)$$

We define the recurrence relation:

$$\begin{cases} y_0 = g, \\ y_1 = N(y_0) \\ y_{m+1} = N(y_0 + \ldots + y_m) - N(y_0 + \ldots + y_{m-1}), m = 1,2,\ldots \end{cases} \quad \ldots(5)$$

Then

$$(y_1 + \ldots + y_{m+1}) = N(y_0 + \ldots + y_m), m = 1,2,\ldots \quad \ldots(6)$$

and

$$y = g + \sum_{i=0}^{\infty} y_i \,. \quad \ldots(7)$$

and the series $\sum_{i=0}^{\infty} y_i$ absolutely and uniformly converges to a solution of Eqn. (1).

Effective Algorithm for Solving Nonlinear Fractional Delay Differential Equations

In this section, we present an appropriate algorithm for solving Nonlinear Fractional Delay Differential Equations (FDDEs) using the proposed New Iterative Method (NIM). Consider the following form of Nonlinear Fractional Delay Differential Equations:

$$\begin{cases} D^{\alpha} y(x) + Ly(x) + N\left[y\left(\frac{x}{2}\right)\right] = g(x), \quad x > 0, \\ y^{(i)} = \delta_i, \qquad i = 0,1,2,\ldots \end{cases} \quad \ldots(8)$$

where L is a linear operator, N, represent a nonlinear operator, $g(x)$ is the source term, and D^{α} is the Caputo fractional derivative of order with $m-1<\alpha-m$. To solve Eqn. (8) by means of the proposed modification of the NIM, we apply the operator J^{α}, the inverse of the operator D^{α}, to both sides of Eqn. (8) as follows:

$$y(x) = \sum_{i=0}^{m-1} y^{(i)}(0)\frac{x^i}{i!} + J^{\alpha}\left[-Ly(x) - Ny\left(\frac{x}{2}\right) + g(x)\right]. \quad \ldots(9)$$

Let's consider dividing this equation into two separate parts as follows:

$$y(x) = N(y(x)) + \sum_{i=0}^{m-1} y^{(i)}(0)\frac{x^i}{i!} \quad \ldots(10)$$

where

$$N(y(x)) = J^{\alpha}\left[g(x) - Ly(x) - Ny\left(\frac{x}{2}\right)\right] \qquad ...(11)$$

In our quest for a solution to Eqn. (10), we seek a representation in the form of a series:

$$y(x) = \sum_{i=0}^{\infty} y_i(x). \qquad ...(12)$$

So that, the components y_i will be determined recursively. Moreover the method defines the nonlinear terms $Ny\left(\frac{x}{2}\right)$ by the El-kalla polynomials:

$$Ny\left(\frac{x}{2}\right) = \sum_{n=0}^{\infty} \overline{A}_n \qquad ...(13)$$

where $\overline{A}_n$ are the El-kalla polynomials that can be generated for all forms of nonlinearity as:

$$\overline{A}_n = f(S_n) - \sum_{i=0}^{n-1} A_i \qquad ...(14)$$

where $\overline{A}_n$, are the El-kalla polynomials, $\overline{A}_0$, $\overline{A}_1$, $\overline{A}_2$,…, $f(S_n)$ is the substitution of the summation of dependent variable in the nonlinear term. For example the El-Kalla polynomials of the nonlinear term $y^2(x)$ and the nonlinear term $y^3(x)$ are shown above

El-Kalla polynomials of $y^2(x)$

$$\begin{cases} \overline{A}_0 = y_0^2(x) \\ \overline{A}_1 = 2y_0(x)y_1(x) + y_1^2(x) \\ \overline{A}_2 = 2y_0(x)y_2(x) + 2y_1(x)y_2(x) + y_2^2(x) \\ \overline{A}_3 = 2y_0(x)y_3(x) + 2y_1(x)y_3(x) + 2y_2(x)y_3(x) + y_3^2(x) \\ \overline{A}_4 = 2y_0(x)y_4(x) + 2y_1(x)y_4(x) + 2y_2(x)y_4(x) + 2y_3(x)y_4(x) + y_4^2(x) \end{cases} \qquad ...(15)$$

El-Kalla polynomials of $y^3(x)$

$$\begin{cases} \overline{A}_0 = y_0^3(x) \\ \overline{A}_1 = 3y_1(x)y_0^2(x) + 3y_0(x)y_1^2(x) + y_1^3(x) \\ \overline{A}_2 = 3y_2y_0^2 + 6y_0y_1y_2 + 3y_2y_1^2 + 3y_1y_2^2 + y_2^3 \\ \overline{A}_3 = 3y_3y_0^2 + 6y_0y_1y_3 + 3y_0y_3^2 + 3y_3y_1^2 + 6y_1y_2y_3 + 3y_1y_3^2 + 3y_3y_2^2 + y_3^3 \end{cases} \qquad ...(16)$$

Substituting Eqns. (12) and (13) into Eqn. (9) gives:

$$\sum_{i=0}^{\infty} y_i = \sum_{i=0}^{m-1} y^{(i)}(0)\frac{x^i}{i!} + j^{\alpha}\left(\sum_{n=0}^{\infty} \overline{A}_n\right). \quad ...(17)$$

To determined $y_i(x)$ $i \geq 0$. first we identify the zero component $y_0(x)$ by the terms $\sum_{i=0}^{m-1} y^{(i)}(0)\frac{x^i}{i!}$ and $j^{\alpha}[g(x)]$ where $g(x)$ represents the inhomogeneous terms. Secondly, the remaining components of $y(x)$ can be determined in a way such that each component is determined by using the preceding components. In other words, the method introduces the recursive relation:

$$y_0(x) = \sum_{i=0}^{r} y^{(i)}(0)\frac{x^i}{i!} + j^{\alpha}[f(x)], \quad ...(18)$$

$$y_{n+1}(x) = J^{\alpha}\overline{A}_n \qquad ,n \geq 0 \quad ...(19)$$

Example 1 [see (Srivastava, 2020)]. Consider a first-order nonlinear FDDE:

$$y^{\alpha}(x) = 1 - 2y^2\left(\frac{x}{2}\right) \quad 0 < \alpha \leq 1, \qquad 0 \leq x \leq 1, \qquad y(0) = 0.$$

...(20) The analytical solution is given by $y(x) = \sin(x)$

In view of **section 2.2**, the Eqn. (20) is approximately expressed as follows:

$$y(x) = J^{\alpha}\left[1 - 2y^2\left(\frac{x}{2}\right)\right] + \sum_{i=0}^{r} y^{(i)}(0)\frac{x^i}{i!} \quad ...(21)$$

We deduce the following recurrence relation from section 2.2

$$y_0(x) = \sum_{i=0}^{r} y^{(i)}(0)\frac{x^i}{i!} = \frac{1}{\Gamma(\alpha+1)}x^{\alpha}$$

$$y_0\left(\frac{x}{2}\right) = \frac{1}{2\Gamma(\alpha+1)}x^{\alpha}$$

$$\overline{A}_0 = y_0^2\left(\frac{x}{2}\right)$$

$$y^2{}_0\left(\frac{x}{2}\right) = \frac{1}{4\Gamma(\alpha+1)^2}x^{2\alpha}$$

$$y_1(x) = J^{\alpha}\left[-2y_0{}^2\left(\frac{x}{2}\right)\right] = -2J^{\alpha}\left[\frac{1}{4\Gamma(\alpha+1)^2}x^{2\alpha}\right] = -J^{\alpha}\left[\frac{1}{2\Gamma(\alpha+1)^2}x^{2\alpha}\right] = -\frac{\Gamma(2\alpha+1)}{2\Gamma(1+\alpha)^2\Gamma(3\alpha+1)}x^{3\alpha}$$

$$y_1\left(\frac{x}{2}\right) = -\frac{1}{2}\frac{\Gamma(2\alpha+1)\,x^{3\alpha}}{2^{3\alpha}\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}$$

$$\bar{A}_1 = 2y_0\left(\frac{x}{2}\right)y_1\left(\frac{x}{2}\right) + y_1^2\left(\frac{x}{2}\right) =$$

$$-\frac{1}{2}\frac{x^{\alpha}\Gamma(2\alpha+1)\,x^{3\alpha}}{\Gamma(\alpha+1)^3\,2^{3\alpha}\Gamma(3\alpha+1)} + \frac{1}{4}\frac{\Gamma(2\alpha+1)^2\left(x^{3\alpha}\right)^2}{\left(2^{3\alpha}\right)^2\Gamma(\alpha+1)^4\Gamma(3\alpha+1)^2}$$

$$y_2(x) = J^{\alpha}\left[-2\left(2y_0\left(\frac{x}{2}\right)y_1\left(\frac{x}{2}\right) + y_1^2\left(\frac{x}{2}\right)\right)\right] =$$

$$\frac{\Gamma(4\alpha+1)\,\Gamma(2\alpha+1)\,x^{5\alpha}}{\Gamma(5\alpha+1)\,\Gamma(\alpha+1)^3\,2^{3\alpha}\Gamma(3\alpha+1)} - \frac{1}{2}\frac{\Gamma(6\alpha+1)\,\Gamma(2\alpha+1)^2\,x^{7\alpha}}{\left(2^{3\alpha}\right)^2\Gamma(7\alpha+1)\,\Gamma(\alpha+1)^4\,\Gamma(3\alpha+1)^2}$$

Now, the solution of Example 1 is

$$y(x) = y_0(x) + y_1(x) + y_2(x) + \ldots =$$

$$\frac{x^{\alpha}}{\Gamma(\alpha+1)} - \frac{1}{2}\frac{\Gamma(2\alpha+1)\,x^{3\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} + \frac{\Gamma(4\alpha+1)\,\Gamma(2\alpha+1)\,x^{5\alpha}}{\Gamma(5\alpha+1)\,\Gamma(\alpha+1)^3\,2^{3\alpha}\Gamma(3\alpha+1)} - \frac{1}{2}\frac{\Gamma(6\alpha+1)\,\Gamma(2\alpha+1)^2\,x^{7\alpha}}{\left(2^{3\alpha}\right)^2\Gamma(7\alpha+1)\,\Gamma(\alpha+1)^4\,\Gamma(3\alpha+1)^2} \quad \ldots(22)$$

RESULTS AND DISCUSSION

Table 1: Two-term approximate solution by MNIM of **Example 1** using different values of α with a comparison with the exact solution when $\alpha = 1$.

x	EXACT (α=1)	MNIM (α=1)	MNIM (α=0.9)	MNIM (α=0.7)	MNIM (α=0.5)
0	0	0	0	0	0
0.1	0.099833417	0.099833417	0.130464313	0.216891413	0.342398528
0.2	0.198669331	0.198669332	0.241458536	0.345317312	0.465800175
0.3	0.295520207	0.295520223	0.343495344	0.447523039	0.550245715
0.4	0.389418342	0.389418463	0.437791463	0.532347546	0.614569117
0.5	0.479425533	0.479426115	0.524532131	0.603752147	0.666604261
0.6	0.564642446	0.564644529	0.603609802	0.664125393	0.710641517
0.7	0.644217577	0.644223704	0.674836849	0.715174224	0.749381361
0.8	0.717355723	0.717371327	0.738029907	0.758258653	0.784669077
0.9	0.783325850	0.783361437	0.793049828	0.794541806	0.817832138
1	0.841468254	0.841542659	0.839822758	0.825064796	0.849856794

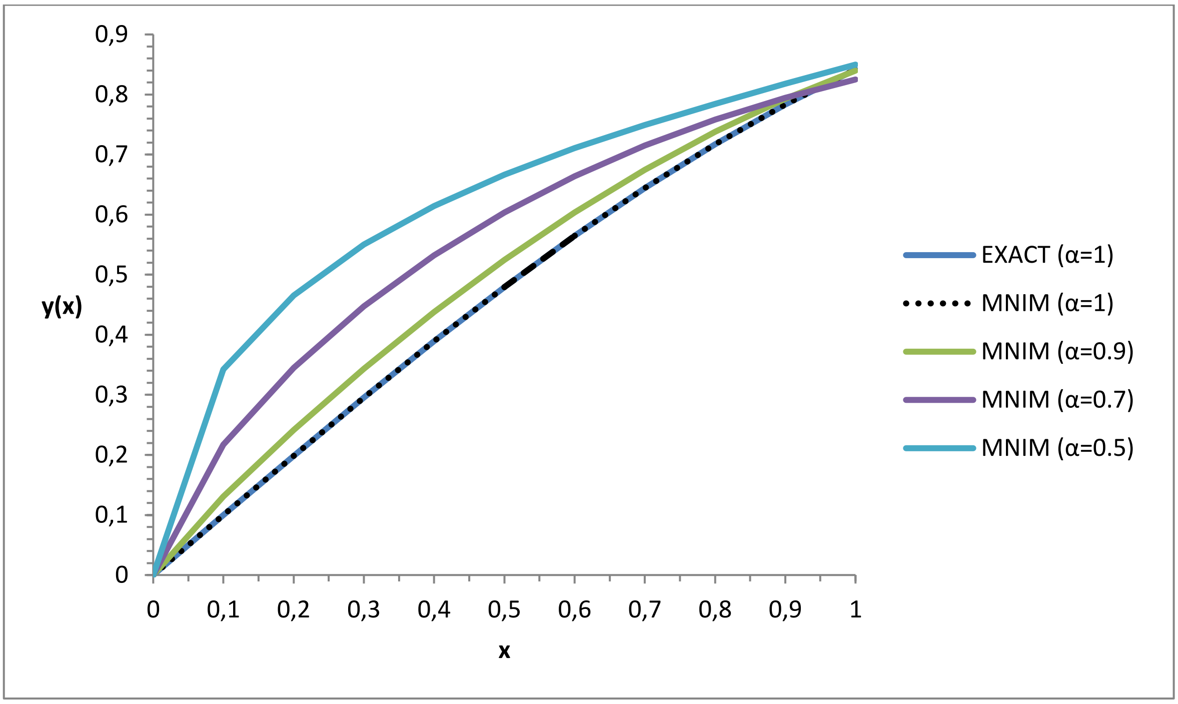

FIGURE 1: Approximate Solution obtained for different values α and the exact solution for α=1 of example 1.

Discussion of Results for Example 1

In this section, we analyze the results of Example 1, which involves the first-order nonlinear Fractional Differential Equation (FDDE) presented in the problem. The goal is to evaluate the accuracy of the method, the Modified New Iterative Method (MNIM) when applied to this example, and how it compares to the exact analytical solution. Table 1 provides the numerical solutions obtained for different values of the parameter α, alongside the exact solution for $\alpha = 1$, at various values of x.

Comparison between MNIM (Approximate Solution) and Exact Solution

At $x = 0$, both the exact and the MNIM solutions across all values of α are zero, which is expected since the initial conditions for the problem are set at zero. As x increases, the differences between the MNIM solutions and the exact solution become more noticeable, especially as α decreases.

- For $\alpha = 1$: The MNIM solution closely matches the exact solution across all values of x, with deviations being minimal throughout the range of x. At $x = 1$, the solutions are practically identical, which confirms the reliability and accuracy of the MNIM when $\alpha = 1$. This indicates that the method works well for this particular case.

- For $\alpha = 0.9$: The MNIM solution still provides a good approximation to the exact solution, but as we move further from $x = 0$, the difference between the two solutions becomes slightly more noticeable. For instance, at $x = 0.1$, the MNIM value is 0.130464313 compared to the exact solution of 0.099833417. The discrepancy grows as x increases, and by $x = 1$, the MNIM solution is 0.839822758, compared to the exact solution of 0.841468254. However, the difference is still relatively small.

- For $\alpha = 0.7$ and smaller values of α: As the value of α decreases further, the differences between the MNIM solutions and the exact solution become more significant. At $\alpha = 0.7$, we observe that the MNIM solution deviates more from the exact solution, particularly as x increases. At $x = 0.1$, for example, the MNIM solution is 0.216891413, while the exact solution is 0.099833417. This trend continues as α decreases further. At $x = 1$, the MNIM result (0.825064796) deviates noticeably from the exact solution (0.841468254).

Impact of Parameter α on Accuracy

The results demonstrate that the value of α plays a critical role in the accuracy of the MNIM approximation. As α decreases, the difference between the MNIM solutions and the exact solution becomes more pronounced. This suggests that the method's performance is more sensitive to smaller values of α, which may be due to the nonlinear nature of the problem and the increasing complexity of the fractional differential equation as α becomes smaller.

It is worth noting that for higher values of α (closer to 1), the MNIM method performs very well and provides solutions that are almost indistinguishable from the exact solution. However, for smaller values of α, the method begins to show a greater level of approximation error. This indicates that while MNIM is a powerful method for solving FDDEs, its accuracy can degrade when dealing with smaller values of α, possibly due to the nature of the recurrence relation derived in Section 2.2, which might not capture the dynamics of the solution as well for these values.

Graphical Representation

Figure 1 visually confirms these findings. The graph compares the MNIM solutions for different values of α against the exact solution for $\alpha = 1$. As expected, for ($\alpha = 1$), the MNIM curve closely follows the exact solution. For smaller values of α, the MNIM solution begins to diverge from the exact solution as x increases. This is most apparent in the curve for $\alpha = 0.5$, where the discrepancy between the MNIM and exact solutions is most pronounced.

The results from Example 1 demonstrate that the MNIM method provides a highly accurate solution for the FDDE when α is close to 1. As α decreases, the accuracy of the MNIM approximation diminishes, which highlights the sensitivity of the method to the parameter α. The method's effectiveness is particularly notable for higher values of α, making it a reliable choice for solving FDDEs in such cases. However, further refinement of the method or alternative approaches may be necessary to improve accuracy for smaller values of α.

Example 2 [see (Moltot & Deresse, 2022)]. Consider the following nonlinear second-order FDDE:

$$D^{\alpha} y(x) = y^2\left(\frac{x}{2}\right) \quad , \qquad x \ge 0, \quad y(0) = 1, \qquad y'(0) = 1. \qquad ...(23)$$

The analytical solution is given by $y(x) = e^x$

Applying the fractional integral operator to both sides of Eqn. (23)

$$y(x) = 1 + x + J^{\alpha}\left[y^2\left(\frac{x}{2}\right)\right] \qquad ...(24)$$

We deduce the following recurrence relation from section 2.2

$$y_0(x) = 1 + x$$

$$y_0\left(\frac{x}{2}\right) = 1 + \frac{x}{2}$$

$$\overline{A}_0 = y_0^2\left(\frac{x}{2}\right)$$

$$y^2{}_0\left(\frac{x}{2}\right) = \left(1 + \frac{x}{2}\right)^2$$

$$y_1(x) = J^{\alpha}\left[y_0{}^2\left(\frac{x}{2}\right)\right] = J^{\alpha}\left[\left(1 + \frac{x}{2}\right)^2\right] = J^{\alpha}\left[1 + x + \frac{x^2}{4}\right]$$

$$= \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{x^{\alpha+3}}{2\Gamma(\alpha+3)}$$

$$y_1\left(\frac{x}{2}\right)=\frac{x^{\alpha}}{\Gamma(\alpha+1)\,2^{\alpha}}+\frac{x^{\alpha+1}}{\Gamma(\alpha+2)\,2^{\alpha+1}}+\frac{1}{2}\frac{x^{\alpha+2}}{\Gamma(\alpha+3)\,2^{\alpha+2}}$$

$$y_2(x)=J^{\alpha}\left[\bar{A}_1\left(\frac{x}{2}\right)\right]=$$

$$\frac{2x^{2\alpha}}{\Gamma(2\alpha+1)\,2^{\alpha}}+\frac{2x^{2\alpha+1}}{\Gamma(2\alpha+2)\,2^{\alpha+1}}+\frac{x^{2\alpha+2}}{\Gamma(2\alpha+3)\,2^{\alpha+2}}+\frac{\Gamma(\alpha+2)\,x^{2\alpha+1}}{\Gamma(2\alpha+2)\,\Gamma(\alpha+1)\,2^{\alpha}}+\frac{\Gamma(\alpha+3)\,x^{2\alpha+2}}{\Gamma(2\alpha+3)\,\Gamma(\alpha+2)\,2^{\alpha+1}}+\frac{1}{2}\frac{\Gamma(\alpha+4)\,x^{2\alpha+3}}{\Gamma(2\alpha+4)\,\Gamma(\alpha+3)\,2^{\alpha+2}}$$

Now, the solution of Example 2 is

$$y(x)=y_0+y_1+y_2+\ldots=$$

$$y(x)=1+x+\frac{x^{\alpha}}{\Gamma(\alpha+1)}+\frac{x^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{1}{2}\frac{x^{\alpha+2}}{\Gamma(\alpha+3)}+\frac{2x^{2\alpha}}{\Gamma(2\alpha+1)\,2^{\alpha}}+\frac{2x^{2\alpha+1}}{\Gamma(2\alpha+2)\,2^{\alpha+1}}+\frac{x^{2\alpha+2}}{\Gamma(2\alpha+3)\,2^{\alpha+2}}+\frac{\Gamma(\alpha+2)\,x^{2\alpha+1}}{\Gamma(2\alpha+2)\,\Gamma(\alpha+1)\,2^{\alpha}}+\frac{\Gamma(\alpha+3)\,x^{2\alpha+2}}{\Gamma(2\alpha+3)\,\Gamma(\alpha+2)\,2^{\alpha+1}}+\frac{1}{2}\frac{\Gamma(\alpha+4)\,x^{2\alpha+3}}{\Gamma(2\alpha+4)\,\Gamma(\alpha+3)\,2^{\alpha+2}} \quad \ldots(25)$$

Table 2: The approximate solution of **Example 2** using SIM and MNIM in comparison with the exact solution when $\alpha=2$.

x	EXACT (α=2)	MNIM (α=2; n=2)	SIM (α=2; n=4)
0	1	1	1
0.1	1.105170918	1.105170917	1.105170833
0.2	1.221402758	1.221402717	1.221400000
0.3	1.349858808	1.349858326	1.349837500
0.4	1.491824698	1.491821917	1.491733333
0.5	1.648721271	1.648710366	1.648437500
0.6	1.822118800	1.822085318	1.821400000
0.7	2.013752707	2.013665883	2.012170833
0.8	2.225540928	2.225341968	2.222400000
0.9	2.459603111	2.459188266	2.453837500
1	2.718281828	2.717478919	2.708333333

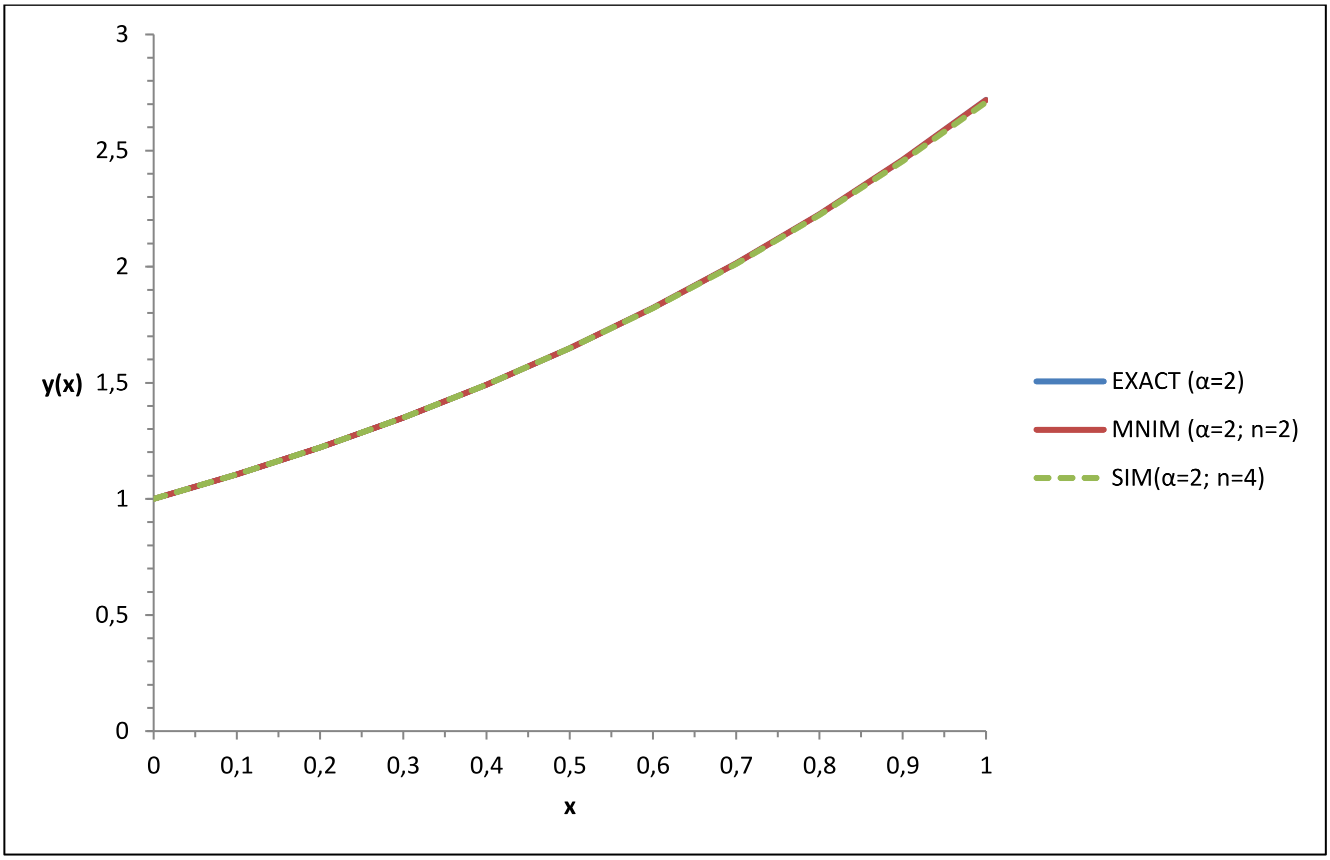

FIGURE 2: Plots of approximate solution obtained using SIM and MNIM in comparison with the exact solution when $\alpha = 2$ of example 2.

Table 3: The approximate solution of **Example 2** using different values of α with a comparison with the exact solution when $\alpha = 2$.

x	EXACT (α=2)	MNIM (α=2; n=2)	MNIM (α=1.8; n=2)	MNIM (α=1.6; n=2)	MNIM (α=1.4; n=2)	MNIM (α=1.2; n=2)
0	1	1	1	1	1	1
0.1	1.105170918	1.105170917	1.109806939	1.118311409	1.133671835	1.161127543
0.2	1.221402758	1.221402717	1.235474491	1.258010394	1.293779576	1.350625191
0.3	1.349858808	1.349858326	1.376540709	1.416191900	1.474997845	1.563185986
0.4	1.491824698	1.491821917	1.533728090	1.593027562	1.677333459	1.799449110
0.5	1.648721271	1.648710366	1.708204329	1.789549228	1.901964703	2.061079435
0.6	1.822118800	1.822085318	1.901439967	2.007268002	2.150641001	2.350200202
0.7	2.013752707	2.013665883	2.115160907	2.248043263	2.425487923	2.669224345
0.8	2.225540928	2.225341968	2.351331783	2.514027447	2.728926190	3.020788747

0.9	2.459603111	2.459188266	2.612152534	2.807641294	3.063633975	3.407725099
1.0	2.718281828	2.717478919	2.900061646	3.131563620	3.432528794	3.833046428
1.1	3.004166024	3.002702863	3.217743244	3.488728843	3.838759403	4.299941205
1.2	3.320116923	3.317579886	3.568136637	3.882329072	4.285703268	4.811771498
1.3	3.669296668	3.665077394	3.954447557	4.315819068	4.776967377	5.372073423
1.4	4.055199967	4.048427919	4.380160680	4.792923134	5.316391183	5.984558995
1.5	4.481689070	4.471147374	4.849053178	5.317643387	5.908050960	6.653118869
1.6	4.953032424	4.937054070	5.365209135	5.894269061	6.556265161	7.381825674
1.7	5.473947392	5.450288524	5.933034729	6.527386623	7.265600514	8.174937760
1.8	6.049647464	6.015334054	6.557274123	7.221890561	8.040878663	9.036903239
1.9	6.685894442	6.637038191	7.243026006	7.982994732	8.887183257	9.972364257
2.0	7.389056099	7.320634921	7.995760769	8.816244220	9.809867395	10.98616143

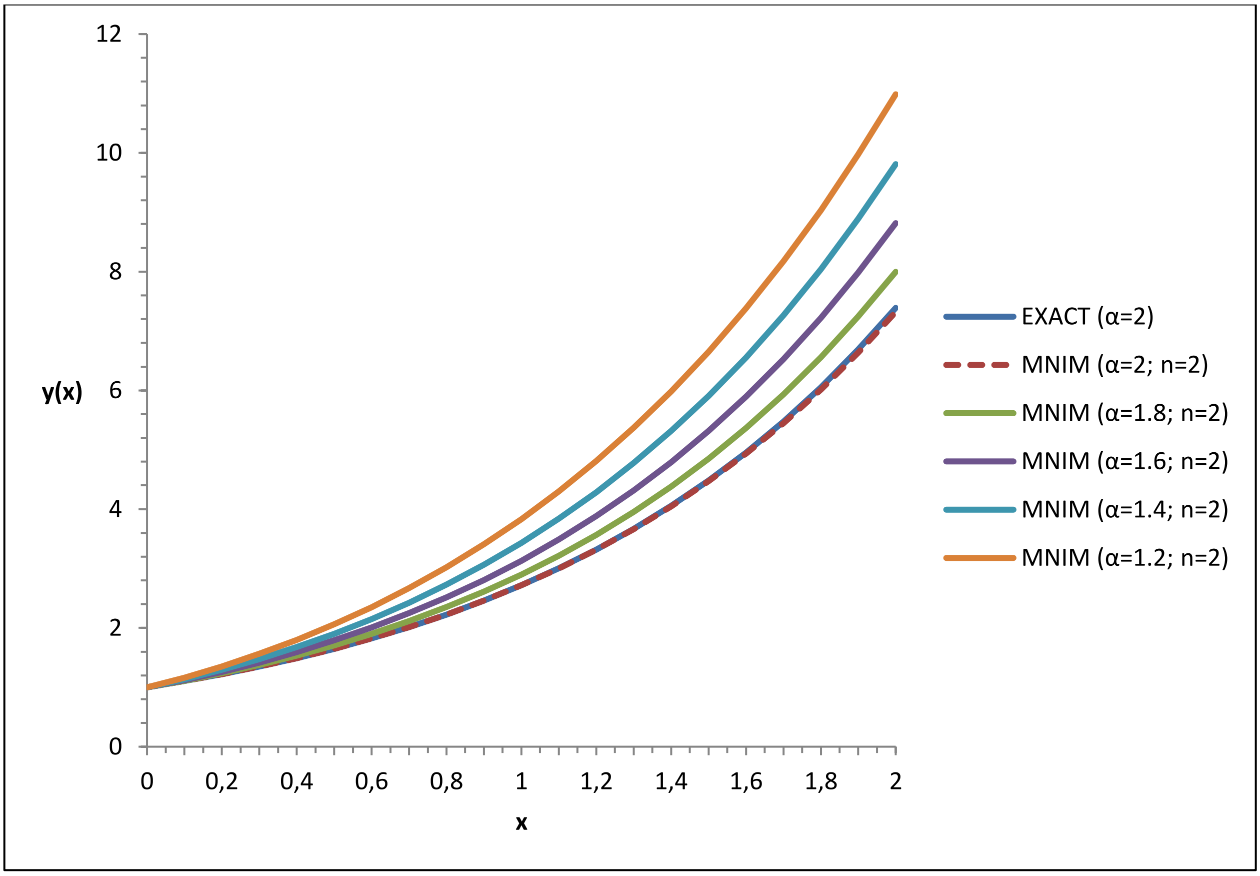

FIGURE 3: Approximate Solution obtained for different values α and the exact solution for α=2 of example 2.

Discussion of Results for Example 2

The approximate solutions for Example 2 obtained using the Sumudu Iterative Method (SIM) and the Modified New Iterative Method (MNIM) were compared with the exact solution to assess the accuracy and efficiency of these numerical methods.

Comparison of SIM and MNIM with the Exact Solution

Table 2 presents the approximate solutions for Example 2 using SIM and MNIM at $\alpha = 2$. The results indicate that both methods provide close approximations to the exact solution, with slight variations. Notably:

- MNIM (α=2, n=2) closely follows the exact solution with minimal deviations across all x values.
- SIM (α=2, n=4) shows slightly more deviation from the exact solution, particularly for higher x values.
- The error increases as x increases, with a more noticeable deviation at x=1, where MNIM (α=2, n=2) gives 2.717478919 compared to the exact solution of 2.718281828, while SIM (α=2, n=4) gives 2.708333333.

Figure 2 illustrates the comparative accuracy of SIM and MNIM. The curves show that both methods align closely with the exact solution, but MNIM tends to provide more precise approximations.

Effect of Changing α on MNIM Accuracy

Table 3 demonstrates the impact of varying fractional order α on the approximate solutions obtained using MNIM. The key observations include:

- As α decreases from 2 to 1.2, the approximate solution diverges further from the exact solution.
- For α=1.8 and α=1.6, the results are still relatively close to the exact solution but show a noticeable increase in deviation as x increases.
- For lower α values (e.g., α=1.4 and α=1.2), the solutions significantly deviate from the exact values, indicating that the accuracy decreases as α moves away from 2.

Figure 3 visualizes these results, showing the increasing deviation from the exact solution as α decreases. The trend confirms that the fractional order plays a crucial role in determining the accuracy of the numerical approximations.

Overall Insights

1. **Method Comparison**: MNIM provides more accurate approximations than SIM, particularly for α=2.
2. **Effect of α**: The accuracy of MNIM decreases as α moves away from 2, highlighting the sensitivity of the method to fractional order changes.
3. **Computational Efficiency**: Both methods yield reliable results, but MNIM appears to be more stable across different x values.

These findings emphasize the importance of selecting appropriate numerical methods and fractional orders for solving nonlinear fractional differential-difference equations (FDDEs). Future studies can explore optimization techniques to enhance accuracy and computational efficiency further.

Example 3 [see (Moltot & Deresse, 2022)]. Consider the following nonlinear first-order FDDE:

$$D^{\alpha} y(x) - 2xy^{4}\left(\frac{x}{2}\right) = 0 \quad , \quad x \ge 0, \quad y(0) = 1. \qquad ...(26)$$

The analytical solution is given by $y(x) = e^{x^2}$

Applying the fractional integral operator to both sides of Eqn. (26)

$$y(x) = 1 + J^{\alpha}\left[2xy^{4}\left(\frac{x}{2}\right)\right] \qquad ...(27)$$

We deduce the following recurrence relation from section 2.2

$$y_0(x) = 1$$

$$y_0\left(\frac{x}{2}\right) = 1$$

$$\overline{A}_0 = y_0^4\left(\frac{x}{2}\right) = 1$$

$$y_1(x) = J^{\alpha}\left[2xy_0^{\ 4}\left(\frac{x}{2}\right)\right] = J^{\alpha}[2x] = \frac{2x^{\alpha+1}}{\Gamma(\alpha+2)}$$

$$y_1\left(\frac{x}{2}\right)=\frac{2x^{\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+2)}$$

$$y_2(x)=J^{\alpha}\left[2x\bar{A}_1\left(\frac{x}{2}\right)\right]=\frac{16\,\Gamma(\alpha+3)\,x^{2\alpha+2}}{\Gamma(2\alpha+3)\,\Gamma(\alpha+2)\,2^{\alpha+1}}$$

$$y_2\left(\frac{x}{2}\right)=\frac{16\,\Gamma(\alpha+3)\,x^{2+2\alpha}}{\Gamma(2\alpha+3)\,\Gamma(\alpha+2)\,2^{3\alpha+3}}$$

$$y_3(x)=J^{\alpha}\left[2x\bar{A}_2\left(\frac{x}{2}\right)\right]=$$

$$\frac{128\,\Gamma(2\alpha+4)\,\Gamma(\alpha+3)\,x^{3\alpha+3}}{\Gamma(3\alpha+4)\,\Gamma(2\alpha+3)\,\Gamma(\alpha+2)\,2^{3\alpha+3}}+\frac{48\,\Gamma(2\alpha+4)\,x^{3\alpha+3}}{\Gamma(3\alpha+4)\,\Gamma(\alpha+2)^2\,(2^{2\alpha+2})}$$

Now, the solution of Example 3 is

$$y(x)=y_0+y_1+y_2+y_3+\ldots=$$

$$y(x)=1+\frac{2x^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{16\,\Gamma(\alpha+3)\,x^{2+2\alpha}}{\Gamma(2\alpha+3)\,\Gamma(\alpha+2)\,2^{\alpha+1}}+\frac{128\,\Gamma(2\alpha+4)\,\Gamma(\alpha+3)\,x^{3\alpha+3}}{\Gamma(3\alpha+4)\,\Gamma(2\alpha+3)\,\Gamma(\alpha+2)\,2^{3\alpha+3}}+\frac{48\,\Gamma(2\alpha+4)\,x^{3\alpha+3}}{\Gamma(3\alpha+4)\,\Gamma(\alpha+2)^2\,2^{2+2\alpha}} \quad \ldots(28)$$

TABLE 4: The approximate solution of **Example 3** using different values of α with a comparison with the exact solution when $\alpha=1$.

x	EXACT (α=1)	MNIM (α=1; n=3)	MNIM (α=0.8; n=3)	MNIM (α=0.6; n=3)	MNIM (α=0.4; n=3)	MNIM (α=0.2; n=3)
0.0	1	1	1	1	1	1
0.1	1.010050167	1.010050167	1.019150853	1.036281473	1.069288836	1.137899856
0.2	1.040810774	1.040810667	1.068855126	1.117472200	1.208469372	1.406372500
0.3	1.094174284	1.094171500	1.149988238	1.246162347	1.432877983	1.865554665
0.4	1.173510871	1.173482667	1.268479536	1.436361606	1.778375449	2.601994182
0.5	1.284025417	1.283854167	1.434244915	1.709948058	2.294347291	3.718462336
0.6	1.433329415	1.432576000	1.661612073	2.097046474	3.043410044	5.331867684

0.7	1.632316220	1.629658167	1.970102048	2.636946951	4.101717749	7.572073404
0.8	1.896480879	1.888490667	2.385393555	3.379172273	5.559369230	10.58105416
0.9	2.247907987	2.226623500	2.940411568	4.384591028	7.520816172	14.51223664
1.0	2.718281828	2.666666667	3.676512975	5.726540946	10.10524835	19.52995820

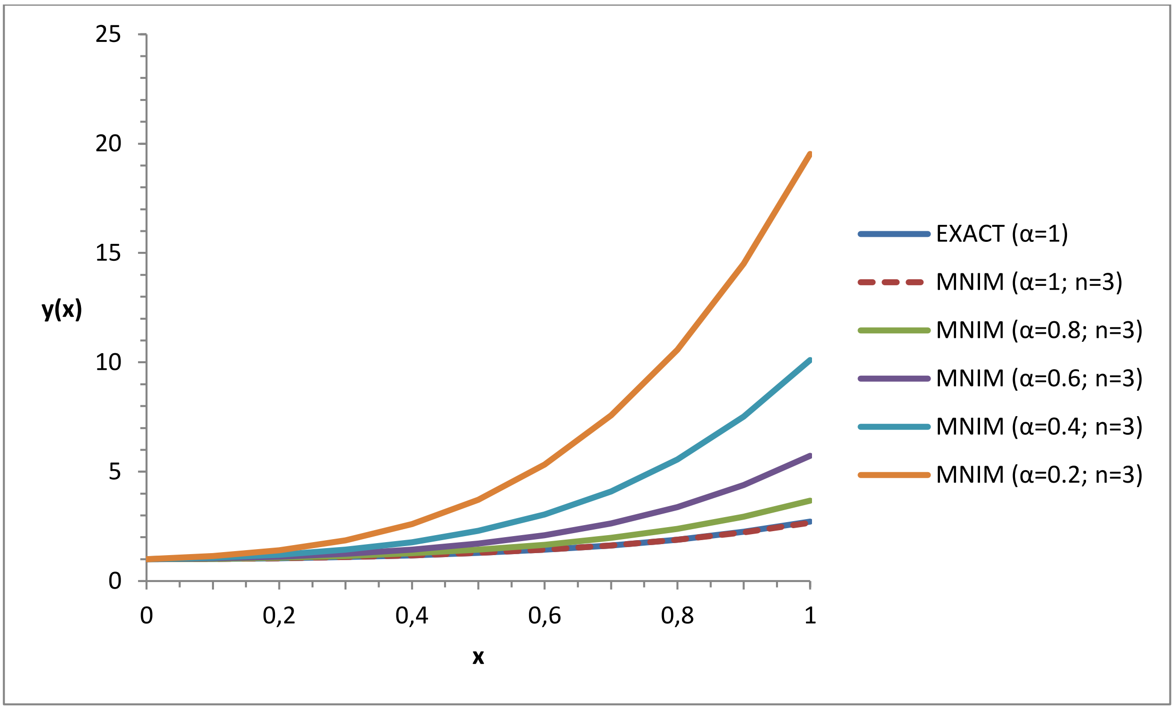

FIGURE 4: Approximate Solution obtained for different values α and the exact solution for α=1 of example 3.

Discussion of Results from Example 3

Example 3 involves a nonlinear first-order fractional differential delay equation (FDDE), and the approximate solutions obtained using the Modified New Iterative Method (MNIM) are compared with the exact solution. In this case, we explore how the fractional order α affects the accuracy of the approximate solutions.

The data in Table 4 show the exact solution and the results from MNIM for several values of α. The solutions are compared for different values of α, including (α = 1, 0.8, 0.6, 0.4, 0.2), while Figure 4 presents a graphical comparison of the approximate solutions for these values of α.

1. Accuracy of MNIM for Different Values of α

From Table 4, we see that the accuracy of the MNIM method varies depending on the value of the fractional order α. For (α = 1), which represents the standard case (without fractional derivatives), the MNIM approximation closely matches the exact solution, especially at smaller values of x. For example, at (x = 0.1), both the exact and approximate solutions are (1.010050167), indicating very high accuracy. As x increases, the approximations for (α = 1) remain quite accurate.

However, as we decrease the value of α (e.g., (α = 0.8, 0.6, 0.4, 0.2)), we observe an increasing difference between the approximate and exact solutions, particularly at larger values of x. For instance, when (α = 0.2), at (x = 1.0), the MNIM solution is (19.52995820), while the exact value is (2.718281828). This highlights a clear trend: as α decreases, the discrepancy between the approximate and exact solutions grows.

2. Influence of Decreasing α on Solution Behavior

The impact of decreasing α is clearly visible in both the numerical results and the graphical plots (Figure 4). As α decreases, the values of the approximate solutions diverge more significantly from the exact solution, particularly in the later stages of the domain $(x \to 1)$.

At lower fractional orders (α < 1), the rate of growth of the solution becomes more pronounced, as seen in the values of ($y(x)$) for (α = 1.0). For example, when (α = 0.2), the value of $y(0.1)$ is significantly higher compared to the exact solution, which suggests that the fractional derivative introduces a compounding effect on the system that leads to a larger growth rate.

This result indicates that smaller fractional orders lead to solutions that grow at a faster rate. The system's behavior becomes more sensitive to changes in the fractional order, making it important to carefully choose α for specific applications. The increasing discrepancy between the MNIM approximations and the exact solution as α decreases also reflects that fractional order plays a critical role in determining the solution's behavior.

3. Visual Analysis of Approximate Solutions

In Figure 4, the plots of the exact and approximate solutions for various values of α illustrate the impact of fractional order on the solution's behavior. The curve for (α = 1) is closest to the exact solution, indicating high accuracy. As α decreases, the approximate solutions diverge more significantly from the exact curve, especially as x increases towards 1.

4. Summary of Observations

- For ($\alpha = 1$), MNIM provides highly accurate results, and the approximation closely follows the exact solution across the entire range of x.

- The growth behavior of the solution is more pronounced for lower values of α, indicating that the system becomes more sensitive to the fractional order.

- The error increases as α decreases, particularly at larger values of x.

The results from Example 3 indicate that the Modified New Iterative Method (MNIM) is highly effective for approximating the solution of a first-order fractional differential delay equation when the fractional order α is close to 1. However, as α decreases, the graph moves away from the accurate solution, particularly for larger values of x. These observations highlight the critical role of the fractional order in determining the accuracy and behavior of the solution, with smaller values of α leading to significant discrepancies between the approximate and exact solutions.

CONCLUSION

The Modified New Iterative Method (MNIM) has demonstrated itself as a reliable and efficient approach for solving nonlinear fractional delay differential equations (NFDDEs). The method consistently yields accurate approximations, particularly when the fractional order is close to an integer. For instance, in Example 2, MNIM closely mirrors the exact solution, maintaining high accuracy throughout the entire domain. Likewise, in Examples 1, 2, and 3, when the fractional order equals the integer order of the respective equations, MNIM provides highly precise results underscoring its capability in handling traditional (integer-order) delay differential equations.

However, the method's accuracy diminishes as the fractional order decreases. This limitation is most evident in Example 3, where lower fractional orders result in larger deviations between the approximate and exact solutions, especially at higher values of the independent variable. The method's reliability decreases as the fractional component becomes more pronounced, revealing its sensitivity to lower fractional orders. When fractional behavior dominates, MNIM struggles to accurately model the system's dynamics. In summary, while MNIM proves to be a powerful and versatile technique for solving fractional differential equations, its performance is best when the fractional order is moderate to high, offering excellent accuracy in such cases.

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