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A New Numerical Scheme for Solving Quadratic Riccati Differential Equations (QRDEs)

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Abstract

Quadratic Riccati Differential Equations (QRDEs) are important in control theory, optimal stabilization, and nonlinear dynamics, thereby requiring numerical methods that are both accurate and computationally reliable. This study introduces a new numerical scheme (NNS) for solving QRDEs using an eighth-order power series basis function combined with interpolation and collocation techniques to approximate the exact solutions over a one-step integration interval. Through this procedure, the continuous differential equation is transformed into a system of nonlinear algebraic equations, which is then solved using the Gauss elimination method. A detailed analysis of the proposed scheme establishes its high order of accuracy, zero stability, consistency, convergence, and absolute stability, confirming its suitability for practical computation. The method was implemented on four different QRDE problems, and the results showed that the approximate solutions were in excellent agreement with the exact solutions throughout the integration interval. The study concludes that the proposed NNS is a robust, accurate, and broadly applicable approach for the numerical solution of QRDEs, offering a reliable contribution to computational methods for nonlinear differential equations.

Keywords: Quadratic Riccati Differential Equations; Numerical Scheme; Power Series Method; Stability Analysis; Convergence

Introduction

The Quadratic Riccati Differential Equations (QRDEs) are one of the most important mathematical tools in different engineering and science disciplines and they are associated with very diverse applications such as robust stabilization, stochastic realization theory, network synthesis, optimal control, and finance among others (Riaz et al., 2015; File & Aga, 2016). Similarly, nonlinear differential equations including the Riccati equation are considered to be the most powerful tools for modeling and simulating a wide range of physical phenomena such as spring-mass systems, R-C-L circuits, beam bending, chemical reactions, pendulum motion, and even rotation (Vahidi & Didgar, 2012).

Riccati Differential Equations (RDE) are a main type of nonlinear differential equations; they have become widely popular in various fields of application in science and technology (Nasr Al-Din, 2020a). Primarily, they have a close connection to the one-dimensional static Schrödinger equation (Nasr Al-Din, 2020b). The equations, named after the Italian nobleman Count Jacopo Francesco Riccati (1676–1754), are not only used in random processes, optimal control, and diffusion problems, but also in stochastic realization theory, optimal control, network synthesis, and financial mathematics (Nasr Al-Din, 2020a).

Nonlinear differential equations are indispensable for modeling the entire spectrum of physical situations like spring-mass systems, resistor-capacitor-inductor circuits, bending of beams, chemical reactions, pendulum motion, and so on (Wase et al., 2021). The Quadratic Riccati Differential Equation (QRDE) is a first-order ordinary differential equation that is usually written in the following form:

$$y'(t) = ay^2(t) + bt + c \tag{1.1}$$

In this study, the dependent variable is and the independent variable is . The quadratic Riccati differential equation is a specific scenario where and are constants and one of the methods commonly applied for solving the quadratic Riccati differential equation is the introduction of a transformation that turns the equation into a linear second-order ordinary differential equation, which can subsequently be solved through the application of standard techniques.

Quadratic Riccati equations are encountered very often in the diverse areas of mathematics and physics like control theory, differential geometry, and mathematical physics. They are used in various areas such as optimal control, stability analysis, and nonlinear dynamics (Sunday, 2017). The representation of solitary wave solutions for a nonlinear partial differential equation generally consists of writing them as a polynomial in two elementary functions that obey a projective Riccati equation (Vinod & Dimple, 2016). This method is not restricted to wave dynamics only but it is also applicable to issues concerning optimal control. This particular topic has therefore attracted a lot of attention and has been deeply studied by many researchers.

Even though the problem is nonlinear, similar to second-order inhomogeneous linear ordinary differential equations, it is still enough to derive the general solution just by acquiring a particular one. The problem has received a lot of attention and a number of researchers have been working on it. Tan & Abbasbandy (2018) applied the Homotopy Analysis Method (HAM) to a quadratic Riccati equation, on the other hand, Sunday & Philip, (2018) proposed a solution for the Riccati equation with variable coefficients through the differential transformation method.

A new approach that uses the multistage variational iteration method has been suggested as a powerful technique for solving quadratic Riccati differential equations (Batiha, 2015). The method consists of applying Legendre scaling functions to develop the approximate solution needed, together with the operational matrix of integral, thus converting the problem into a series of algebraic equations (Baghchehjoughi et al., 2014). Besides, the iterative decomposition algorithm has been used to derive approximations of generalized Riccati differential equations, too.

File and Aga (2016) proposed a fourth-order Runge-Kutta method that specifically addressed the application of quadratic Riccati differential equations. Vinod and Dimple (2016) came up with a hybrid Laplace-Adomian decomposition technique based on the Newton-Raphson method, which the objective was to solve such kinds of equations. Besides, Ghomanjani and Khorram (2015), Ghomanjani and Momani (2020) mentioned a method that collaborates with Bezier curves implied the creation of Bezier polynomials of a specified degree. Additionally, Ghomanjani and Shateyi (2020) presented a powerful algorithm based on Genocchi polynomials for the solution of binary Riccati differential equations in a completely different manner.

The study intends to develop a method that not only improves the existing techniques in terms of accuracy and stability but also considers the ongoing attempts to enhance accuracy in solution methodologies. Yet, the mentioned methods have some restrictions and disadvantages. Among them, one can mention that there are significant differences in the results produced by the differential transformation method (DTM). However, as our examples have shown, the new method we have developed gives numeric results that are very close to the exact values with pointwise absolute errors that are quite small.

This research introduces a new computational technique for addressing the nonlinear quadratic Riccati differential equation (1.1). Also, we carry out a numerical analysis to explore the requirements for the computational technique. We validate the approximate solutions produced by our method against exact solutions for different mesh sizes at particular nodal points. In addition, to showcase the capability of our method, we tackle three numerical examples.

Materials and Method for New Numerical Scheme (NNS)

Within this section, we construct New Numerical Scheme (NNS) for resolving QRDEs, as described by equation (1.1). The primary concept driving this endeavor is to estimate the precise solution of equation (1.1) over the partition,

$\pi_{[a,b]} = \left[a = t_0 < t_1 < t_2 < \ldots < t_n < t_{n+1} < \ldots < t_N = b\right]$, of the one-step integration interval $[a,b]$ by power series basis function given by,

$$y(t) = \sum_{n=0}^{r+s-1} a_n t^n \tag{2.1}$$

where $r \text{ and } s$ are the numbers of collocation and interpolation points respectively.

The derivation of NNM will employ interpolation and collocation techniques. Initially, Equation (2.1) will be interpolated at a designated grid point. Subsequently, the first derivative of (2.1) will replace (1.1) to formulate a differential system, which will be evaluated across all grid points.

Derivation of the New Numerical Scheme (NNS)

Let the approximate solution to (1.1) be given by power series of degree 8, by allowing $r+s-1=8$ in equation (2.1), that is,

$$y(t)=\sum_{n=0}^{8} a_n t^n = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + a_7 t^7 + a_8 t^8 \qquad (2.2)$$

with the first derivative given by,

$$y'(t)=a_1+2a_2t+3a_3t^2+4a_4t^3+5a_5t^4+6a_6t^5+7a_7t^6+8a_8t^7 \qquad (2.3)$$

Substituting (2.3) into (1.1) gives,

$$f(t,y)=a_1+2a_2t+3a_3t^2+4a_4t^3+5a_5t^4+6a_6t^5+7a_7t^6+8a_8t^7 \qquad (2.4)$$

Now, interpolating (2.2) at point $t_{n+s}, s=1$ and collocating (2.4) at points $t_{n+r}=0\left(\frac{1}{7}\right)1$, leads to a system of nonlinear equation of the form

$$TZ=P \qquad (2.5),$$

where

$$Z=\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \end{bmatrix} \quad P=\begin{bmatrix} y_{n+1} & f_n & f_{n+\frac{1}{7}} & f_{n+\frac{2}{7}} & f_{n+\frac{3}{7}} & f_{n+\frac{4}{7}} & f_{n+\frac{5}{7}} & f_{n+\frac{6}{7}} & f_{n+1} \end{bmatrix}^T$$

$$T=\begin{bmatrix}
1 & t_{n+1} & t_{n+1}^2 & t_{n+1}^3 & t_{n+1}^4 & t_{n+1}^5 & t_{n+1}^6 & t_{n+1}^7 & t_{n+1}^8 \\
0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 & 6t_n^5 & 7t_n^6 & 8t_n^7 \\
0 & 1 & 2t_{n+\frac{1}{7}} & 3t_{n+\frac{1}{7}}^2 & 4t_{n+\frac{1}{7}}^3 & 5t_{n+\frac{1}{7}}^4 & 6t_{n+\frac{1}{7}}^5 & 7t_{n+\frac{1}{7}}^6 & 8t_{n+\frac{1}{7}}^7 \\
0 & 1 & 2t_{n+\frac{2}{7}} & 3t_{n+\frac{2}{7}}^2 & 4t_{n+\frac{2}{7}}^3 & 5t_{n+\frac{2}{7}}^4 & 6t_{n+\frac{2}{7}}^5 & 7t_{n+\frac{2}{7}}^6 & 8t_{n+\frac{2}{7}}^7 \\
0 & 1 & 2t_{n+\frac{3}{7}} & 3t_{n+\frac{3}{7}}^2 & 4t_{n+\frac{3}{7}}^3 & 5t_{n+\frac{3}{7}}^4 & 6t_{n+\frac{3}{7}}^5 & 7t_{n+\frac{3}{7}}^6 & 8t_{n+\frac{3}{7}}^7 \\
0 & 1 & 2t_{n+\frac{4}{7}} & 3t_{n+\frac{4}{7}}^2 & 4t_{n+\frac{4}{7}}^3 & 5t_{n+\frac{4}{7}}^4 & 6t_{n+\frac{4}{7}}^5 & 7t_{n+\frac{4}{7}}^6 & 8t_{n+\frac{4}{7}}^7 \\
0 & 1 & 2t_{n+\frac{5}{7}} & 3t_{n+\frac{5}{7}}^2 & 4t_{n+\frac{5}{7}}^3 & 5t_{n+\frac{5}{7}}^4 & 6t_{n+\frac{5}{7}}^5 & 7t_{n+\frac{5}{7}}^6 & 8t_{n+\frac{5}{7}}^7 \\
0 & 1 & 2t_{n+\frac{6}{7}} & 3t_{n+\frac{6}{7}}^2 & 4t_{n+\frac{6}{7}}^3 & 5t_{n+\frac{6}{7}}^4 & 6t_{n+\frac{6}{7}}^5 & 7t_{n+\frac{6}{7}}^6 & 8t_{n+\frac{6}{7}}^7 \\
0 & 1 & 2t_{n+1} & 3t_{n+1}^2 & 4t_{n+1}^3 & 5t_{n+1}^4 & 6t_{n+1}^5 & 7t_{n+1}^6 & 8t_{n+1}^7
\end{bmatrix}$$

Similarly, solving the system of nonlinear equation by Gauss elimination method for the $a_j's,\ j=0(1)8$ and substituting back into the power series basis function gives a new numerical method of the form,

$$y(t)=a_1(t)y_{n+1}+h\sum_{j=0}^{1}\beta_j(t)f_{n+j},\ j=0\left(\frac{1}{7}\right)1 \qquad (2.6)$$

where the coefficients of y_n *and* f_{n+j} are given as,

$$\left.\begin{aligned}
&\alpha_1=1\\
&\beta_0=-\frac{751}{17280}+t-\frac{363}{40}t^2+\frac{22981}{540}t^3-\frac{331681}{2880}t^4+\frac{16807}{90}t^5-\frac{386561}{2160}t^6+\frac{16807}{180}t^7-\frac{117649}{5760}t^8\\
&\beta_{\frac{1}{7}}=-\frac{3577}{17280}+\frac{49}{2}t^2-\frac{10927}{60}t^3+\frac{109417}{180}t^4-\frac{88837}{80}t^5+\frac{991613}{864}t^6-\frac{50421}{80}t^7+\frac{823543}{5760}t^8\\
&\beta_{\frac{2}{7}}=-\frac{49}{640}-\frac{147}{4}t^2+\frac{14357}{40}t^3-\frac{1347647}{960}t^4+\frac{170471}{60}t^5-\frac{50421}{16}t^6+\frac{218491}{120}t^7-\frac{823543}{1920}t^8\\
&\beta_{\frac{3}{7}}=-\frac{2989}{17280}+\frac{245}{6}t^2-\frac{46501}{108}t^3+\frac{133427}{72}t^4-\frac{2926819}{720}t^5+\frac{4151329}{864}t^6-\frac{420175}{144}t^7+\frac{823543}{1152}t^8\\
&\beta_{\frac{4}{7}}=-\frac{2989}{17280}-\frac{245}{8}t^2+\frac{2009}{6}t^3-\frac{872935}{576}t^4+\frac{52822}{15}t^5-\frac{1899191}{432}t^6+\frac{16807}{6}t^7-\frac{823543}{1152}t^8\\
&\beta_{\frac{5}{7}}=-\frac{49}{640}+\frac{147}{10}t^2-\frac{3283}{20}t^3+\frac{22981}{30}t^4-\frac{88837}{48}t^5+\frac{386561}{160}t^6-\frac{386561}{240}t^7+\frac{823543}{1920}t^8\\
&\beta_{\frac{6}{7}}=-\frac{3577}{17280}-\frac{49}{12}t^2+\frac{49931}{1080}t^3-\frac{634207}{2880}t^4+\frac{98441}{180}t^5-\frac{319333}{432}t^6+\frac{184877}{360}t^7-\frac{823543}{5760}t^8\\
&\beta_1=-\frac{751}{17280}+\frac{1}{2}t^2-\frac{343}{60}t^3+\frac{9947}{360}t^4-\frac{16807}{240}t^5+\frac{84035}{864}t^6-\frac{16807}{240}t^7+\frac{117649}{5760}t^8
\end{aligned}\right\} \qquad (2.7)$$

and t is given by

$$x=\frac{t-t_n}{h} \qquad (2.8)$$

Thus, evaluating (2.6) at all points using (2.5), we obtain a set of $r\times r$ matrix schemes which can be written in the form of discrete new numerical method as,

$$A^{(0)}\mathbf{Y}_m=E\mathbf{y}_n+hd\mathbf{f}(\mathbf{y}_n)+hb\mathbf{F}(\mathbf{Y}_m) \qquad (2.9)$$

Evaluating (2.6) at $t=\frac{1}{7}\left(\frac{1}{7}\right)1$, gives the new discrete new numerical method of the form (2.9) given as,

$$
\begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix}
\begin{bmatrix} y_{n+\frac{1}{7}}\\ y_{n+\frac{2}{7}}\\ y_{n+\frac{3}{7}}\\ y_{n+\frac{4}{7}}\\ y_{n+\frac{5}{7}}\\ y_{n+\frac{6}{7}}\\ y_{n+1} \end{bmatrix}
=
\begin{bmatrix} 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1 \end{bmatrix}
\begin{bmatrix} y_{n-\frac{1}{7}}\\ y_{n-\frac{2}{7}}\\ y_{n-\frac{3}{7}}\\ y_{n-\frac{4}{7}}\\ y_{n-\frac{5}{7}}\\ y_{n-\frac{6}{7}}\\ y_{n} \end{bmatrix}
+h
\begin{bmatrix} 0&0&0&0&0&0&\frac{751}{17280}\\ 0&0&0&0&0&0&\frac{41}{980}\\ 0&0&0&0&0&0&\frac{256}{6272}\\ 0&0&0&0&0&0&\frac{278}{6615}\\ 0&0&0&0&0&0&\frac{256}{6272}\\ 0&0&0&0&0&0&\frac{41}{980}\\ 0&0&0&0&0&0&\frac{751}{17280} \end{bmatrix}
\begin{bmatrix} f_{n-\frac{1}{7}}\\ f_{n-\frac{2}{7}}\\ f_{n-\frac{3}{7}}\\ f_{n-\frac{4}{7}}\\ f_{n-\frac{5}{7}}\\ f_{n-\frac{6}{7}}\\ f_{n} \end{bmatrix}
$$

$$
+h\begin{bmatrix}
\frac{139849}{846720} & -\frac{4511}{31360} & \frac{123133}{846720} & -\frac{88547}{846720} & \frac{1537}{31360} & -\frac{11351}{846720} & \frac{275}{169344}\\
\frac{1466}{6615} & -\frac{71}{2940} & \frac{68}{735} & -\frac{1927}{26460} & \frac{26}{735} & -\frac{29}{2940} & \frac{8}{6615}\\
\frac{1359}{6272} & \frac{1377}{31360} & \frac{5927}{31360} & -\frac{3033}{31360} & \frac{1377}{31360} & -\frac{373}{31360} & \frac{9}{6272}\\
\frac{1448}{6615} & \frac{8}{245} & \frac{1784}{6615} & -\frac{106}{6615} & \frac{8}{245} & -\frac{64}{6615} & \frac{8}{6615}\\
\frac{36725}{169344} & \frac{775}{18816} & \frac{4625}{18816} & \frac{13625}{169344} & \frac{1895}{18816} & -\frac{275}{18816} & \frac{275}{169344}\\
\frac{54}{245} & \frac{27}{980} & \frac{68}{245} & \frac{27}{980} & \frac{54}{245} & \frac{41}{980} & 0\\
\frac{3577}{17280} & \frac{49}{640} & \frac{2989}{17280} & \frac{2989}{17280} & \frac{49}{640} & \frac{3577}{17280} & \frac{751}{17280}
\end{bmatrix}
\begin{bmatrix} f_{n+\frac{1}{7}}\\ f_{n+\frac{2}{7}}\\ f_{n+\frac{3}{7}}\\ f_{n+\frac{4}{7}}\\ f_{n+\frac{5}{7}}\\ f_{n+\frac{6}{7}}\\ f_{n+1} \end{bmatrix} \tag{2.10}
$$

Analysis of the Basic properties of the New Numerical Scheme

In this section, the analysis of the basic properties of the NNM shall be carry out. These properties include; order of accuracy and error constant, consistency, root condition, convergence, symmetry and absolute stability.

Order of Accuracy and Error Constant of NNM

Let the linear operator $\ell\{y(t):h\}$ be defined on the NNM, when $j=0$ by the expression

$$
\ell\{y(t):h\} = A^{(0)}\mathrm{Y}_m - \sum_{j=0}^{1}\frac{(ih)^{(j)}}{j!}Ey_n^{(j)} + h\left[d_j f(y_n) + b_j F(\mathrm{Y}_m)\right] \tag{3.1}
$$

It is evident from equation (3.1), that expanding Y_m and $F(\mathrm{Y}_m)$ in Taylor's series and comparing the coefficients of h gives

$$
\ell\{y(t):h\} = \bar{c}_0\, y(t) + \bar{c}_1\, hy'(t) + \cdots + \bar{c}_p\, h^p y^p(t) + \bar{c}_{p+1}\, h^{p+1} y^{p+1}(t) + \cdots \tag{3.2}
$$

Definition 3.1

In numerical analysis, a NNM is a technique for solving ordinary differential equations (ODEs) numerically. The order of a NNM refers to the rate at which the error decreases with decreasing step size. The error constant relates to how quickly the numerical solution approaches the exact solution (Sunday & Philip, 2018).

The order describes the precision of the method in terms of the number of previous steps considered, while the error constant reflects how quickly the method converges to the exact solution. The linear operator ℓ and the associated NNM are said to be of accurate order p if

$$\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \bar{c}_p = \cdots + \bar{c}_{p+1} \neq 0 .$$

The parameter $\bar{c}_{p+1}$ is called the error constant and implies that the truncation error is given by,

$$T_{n+k} = C_{p+1} h^{p+1} y^{p+1} + O\left(h^{p+2}\right) \tag{3.3}$$

$$\begin{bmatrix} \sum_{i=0}^{\infty} \frac{\left(\frac{1}{7}h\right)^i}{i!} \\ \sum_{i=0}^{\infty} \frac{\left(\frac{2}{7}h\right)^i}{i!} \\ \sum_{i=0}^{\infty} \frac{\left(\frac{3}{7}h\right)^i}{i!} \\ \sum_{i=0}^{\infty} \frac{\left(\frac{4}{7}h\right)^i}{i!} \\ \sum_{i=0}^{\infty} \frac{\left(\frac{5}{7}h\right)^i}{i!} \\ \sum_{i=0}^{\infty} \frac{\left(\frac{6}{7}h\right)^i}{i!} \\ \sum_{i=0}^{\infty} \frac{(1h)^i}{i!} \end{bmatrix} - y_n - \begin{bmatrix} -\frac{751}{17280} \\ -\frac{41}{980} \\ -\frac{265}{6272} \\ -\frac{278}{6615} \\ -\frac{265}{6272} \\ -\frac{41}{980} \\ -\frac{751}{17280} \end{bmatrix} hy'_n - \sum_{i=0}^{\infty} \frac{h^{i+1}}{i!n} y_n^{i+1} \begin{bmatrix} \frac{139849}{846720} & -\frac{4511}{31360} & \frac{123133}{846720} & -\frac{88547}{846720} & \frac{1537}{31360} & -\frac{11351}{846720} & \frac{275}{169344} \\ \frac{1466}{6615} & -\frac{71}{2940} & \frac{68}{735} & -\frac{1927}{26460} & \frac{26}{735} & -\frac{29}{2940} & \frac{8}{6615} \\ \frac{1359}{6272} & \frac{1377}{31360} & \frac{5927}{31360} & -\frac{3033}{31360} & \frac{1377}{31360} & -\frac{373}{31360} & \frac{9}{6272} \\ \frac{1448}{6615} & \frac{8}{245} & \frac{1784}{6615} & -\frac{106}{6615} & \frac{8}{245} & -\frac{64}{6615} & \frac{8}{6615} \\ \frac{36725}{169344} & \frac{775}{18816} & \frac{4625}{18816} & \frac{13625}{169344} & \frac{1895}{18816} & -\frac{275}{18816} & \frac{275}{169344} \\ \frac{54}{245} & \frac{27}{980} & \frac{68}{245} & \frac{27}{980} & \frac{54}{245} & \frac{41}{980} & 0 \\ \frac{3577}{17280} & \frac{49}{640} & \frac{2989}{17280} & \frac{2989}{17280} & \frac{49}{640} & \frac{3577}{17280} & \frac{751}{17280} \end{bmatrix} \begin{bmatrix} \left(\frac{1}{7}\right)^i \\ \left(\frac{2}{7}\right)^i \\ \left(\frac{3}{7}\right)^i \\ \left(\frac{4}{7}\right)^i \\ \left(\frac{5}{7}\right)^i \\ \left(\frac{6}{7}\right)^i \\ (1)^i \end{bmatrix} = 0 \tag{3.4}$$

Therefore, if we compare the coefficients of h, the order and the error constant of the method are given by

$$C_9 = \left[-2.3186\times10^{-10}, -1.8203\times10^{-10}, -2.0411\times10^{-10}, -1.8706\times10^{-10}, -2.0914\times10^{-10}, -1.5931\times10^{-10}, -3.9117\times10^{-10}\right]^T$$

respectively. Thus, the NNM is of accurate uniform eight order (Sunday & Philip, 2018).

Root Condition and Zero Stability of the NNM

Definition 3.2

For a NNM to be zero-stable, the roots of the stability polynomial $R(z)$ must lie inside the unit circle in the complex plane. In other words, all roots z of the stability polynomial must satisfy $|z| < 1$. This condition ensures that the numerical solution does not exhibit unbounded growth when applied to the solution of a homogeneous linear ODE with zero forcing term.

Zero stability is a desirable property in numerical methods because it guarantees that small perturbations in the initial conditions do not lead to exponential growth in the numerical solution. It provides confidence that the numerical method will produce stable and reliable results, particularly for long-term simulations or when dealing with stiff differential equations (Sunday, 2017).

The method is stable if and only if all roots of the stability polynomial lie inside the unit circle in the complex plane, i.e., $|z| < 1$. This condition ensures that the numerical solution does not exhibit unbounded growth for any initial condition and step size.

So, the necessary and sufficient condition for the stability of a linear multistep method is:

i. Necessary condition: All roots of the stability polynomial must lie inside the unit circle for the method to be stable.

ii. Sufficient condition: If all roots of the stability polynomial lie inside the unit circle, the method is stable.

This condition guarantees stability for the method for any choice of step size and any initial conditions, making it a fundamental criterion in assessing the stability of numerical methods for solving differential equations.

Let's examine whether the NNM satisfies the root condition,

$$R(z) = \left| z \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right| = 0 \quad (3.5)$$

Solving for z in (3.5), we have

$$\rho(z) = z^6(z-1) = 0$$
$$\Rightarrow z_1 = z_2 = z_3 = z_4 = z_5 = z_6 = 0, z_7 = 1$$

Therefore, the NNM fulfills the root condition (Ayinde, et al. 2023).

Consistency of the NNM

To put it simply, a method's consistency governs the extent of the local truncation error incurred during computation at each step. Therefore, given its uniform order, we can affirm that the NNM is consistent, since $p = 8 \geq 1$ (Ayinde, et al. 2023).

Convergence of the NNM

Theorem 3.1

The Dahlquist theorem 3.1, also known as the convergence theorem, provides conditions under which a NNM for solving ordinary differential equations (ODEs) converges to the exact solution as the step size tends to zero. Formally stated:

"If a linear multistep method is zero-stable and consistent, and if it satisfies a Lipschitz condition with a Lipschitz constant L then the method converges."

As a consequence of the Dahlquist theorem mentioned earlier, NNM can be considered convergent (Ayinde, et al. 2023).

Stability Region of the NNM

Definition

The stability region of a Quadratic Riccati Differential Equation is a concept that arises in control theory. Specifically, it refers to the region in the complex plane for which the solutions of the Riccati equation remain bounded. This is important in analyzing the stability of certain dynamical systems governed by differential equations (Yang et al 2012).

The stability region is typically visualized in the complex plane, with different regions corresponding to different behaviors of the numerical method.

The stability polynomial associated with the new method (2.10) can be expressed as:

$$\bar{h}(w)=\left(-\frac{4507}{2529924096}w^{7}-\frac{69371950261}{187437016424448000}w^{6}\right)h^{7}+\left(-\frac{331024314991}{446278610534400000}w^{6}+\frac{273871}{75897722880}w^{7}\right)h^{6}$$
$$+\left(-\frac{163985805373}{22222}w^{6}-\frac{213337}{553190400}w^{7}\right)h^{5}+\left(-\frac{4118046305983}{2007417323520000}w^{6}+\frac{10030753}{6638284800}w^{7}\right)h^{4}+\left(\frac{579227854099}{133827821568000}w^{6}-\frac{3462563}{135475200}w^{7}\right)h^{3}$$
$$+\left(-\frac{2099940793}{14936140800}w^{6}+\frac{14137079}{135475200}w^{7}\right)h^{2}+\left(\frac{93379}{282240}w^{6}-\frac{1}{2}w^{6}\right)h \tag{3.6}$$

The stability region of NNM is shown figure 3.1 below as

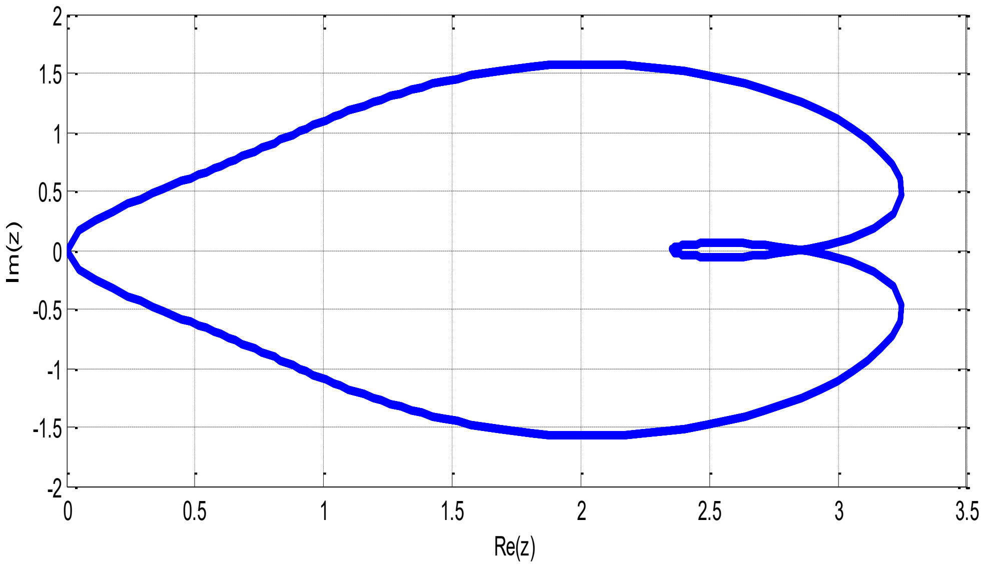

Figure 3.1: Region of absolute stability.

The region of absolute stability in figure 3.1 is $A-stable$.

The A-stable region, also known as the region of absolute stability, is a region in the complex plane where the eigenvalues of the linear part of the ODE solver's discretization matrix lie for the method to remain stable (Sabo et al. 2021). The stable region corresponds to the interior of the curve, whereas the unstable region encompasses the complex plane outside of this enclosed figure 3.1.

Numerical Application of New Numerical Scheme (NNS)

In this section, the numerical application of new numerical scheme was carryout by testing the NNS on some QRDEs of the form (1.1).

The following notations were used in the tables and figures.

AENNS Absolute error in New Numerical Scheme

AES17	Absolute error in Sunday, (2017)
AEFA16	Absolute error in File & Aga, (2016)
AESP18	Absolute error in Sunday & Philip, (2018)

Problem 4.1

Consider the QRE of the form,

$$y'(t) = 10 + 3y(t) - y^2(t), \quad y(0) = 0 \tag{4.1}$$

whose exact solution is given by,

$$y(t) = -2 + \frac{14e^{7t}}{5 + 2e^{7t}} \tag{4.2}$$

Source: [Sunday, (2017), Sunday & Philip, (2018)]

Problem 4.2

Consider the QRE of the form,

$$y'(t) = -\frac{1}{1+t} + y - y^2(t), \quad y(0) = 1 \tag{4.3}$$

whose exact solution is given by,

$$y(t) = \frac{1}{1+t} \tag{4.4}$$

Source: [Sunday, (2017), File & Aga, (2016)]

Problem 4.3

Consider the QRE of the form,

$$y'(t) = 1 - y^2(t), \quad y(0) = 0 \tag{4.5}$$

whose exact solution is given by,

$$y(t) = \frac{e^{2t} - 1}{e^{2t} + 1} \tag{4.6}$$

Source: [Sunday, (2017), Sunday & Philip, (2018)]

Problem 4.4

Consider the QRE of the form,

$$y'(t) = y^2(t) - 1, \quad y(0) = 0 \tag{4.7}$$

whose exact solution is given by,

$$y(t) = -\tanh(t) \tag{4.8}$$

Source: [Sunday, (2017), Sunday & Philip, (2018)].

Table 4.1: Results for Problem 4.1

t	**Analytic Solution**	**Approximate Solution**	**ENNS**	**AES17**	**AESP18**
0.100	1.12295995501998517310	1.12295995587307901600	8.53094E-10	2.82693E-09	1.46347E-09
0.200	2.33036366723934260660	2.33036366744954282610	2.10200E-10	5.89943E-09	2.99223E-09
0.300	3.35929859139218860340	3.35929859155915016660	1.66962E-10	6.83092E-08	3.49315E-08
0.400	4.07625619989394993700	4.07625620020764014650	3.13690E-10	1.49912E-07	7.66512E-08
0.500	4.50864023794231405830	4.50864023797281103320	3.04970E-11	1.83945E-07	9.40192E-08
0.600	4.74705986375186756050	4.74705986386142948480	1.09562E-10	1.65588E-07	8.46132E-08
0.700	4.87206646548954668230	4.87206646560212625910	1.12577E-10	1.24703E-07	6.37095E-08
0.800	4.93588015111826406050	4.93588015119091485170	7.26508E-11	8.43126E-08	4.30686E-08
0.900	4.96801151790818190370	4.96801151794637027740	3.81884E-11	5.32397E-08	2.71935E-08
1.000	4.98407836223863766150	4.98407836225661403400	1.79764E-11	3.21259E-08	1.64080E-08

Table 4.2: Results for Problem 4.2

t	**Analytic Solution**	**Approximate Solution**	**ENNS**	**AES17**	**AEFA16**
0.100	0.90909090909090909091	0.90909090909082528872	8.38022E-14	2.29206E-12	3.82960E-07
0.200	0.83333333333333333333	0.83333333333322152217	1.11812E-13	3.11395E-12	3.82960E-07
0.300	0.76923076923076923077	0.76923076923064908414	1.20147E-13	3.37641E-12	5.79510E-07
0.400	0.71428571428571428571	0.71428571428559287666	1.21409E-13	3.42415E-12	6.81330E-07
0.500	0.66666666666666666667	0.66666666666654638155	1.20285E-13	3.39440E-12	7.33940E-07
0.600	0.62500000000000000000	0.62499999999988140902	1.18591E-13	3.34355E-12	7.60910E-07
0.700	0.58823529411764705882	0.58823529411753001968	1.17039E-13	3.29492E-12	7.74830E-07
0.800	0.55555555555555555556	0.55555555555543965566	1.15900E-13	3.25739E-12	7.82570E-07
0.900	0.52631578947368421053	0.52631578947356894786	1.15263E-13	3.23441E-12	7.87990E-07
1.000	0.50000000000000000000	0.49999999999988485639	1.15144E-13	3.22653E-12	7.93260E-07

Table 4.3: Results for Problem 4.3

t	Analytic Solution	Approximate Solution	ENNS	AES17	AESP18
0.100	0.09966799462495581711	0.09966799462495581510	2.01000E-18	1.14908E-14	9.71445E-17
0.200	0.19737532022490400073	0.19737532022490399719	3.54000E-18	6.71685E-14	8.32667E-17
0.300	0.29131261245159090582	0.29131261245159090165	4.17000E-18	1.83353E-13	0.00000E00
0.400	0.37994896225522488527	0.37994896225522488160	3.67000E-18	3.38618E-13	2.22045E-16
0.500	0.46211715726000975851	0.46211715726000975622	2.29000E-18	4.86111E-13	2.22045E-16
0.600	0.53704956699803528586	0.53704956699803528530	5.60000E-18	5.79870E-13	1.11022E-16
0.700	0.60436777711716349631	0.60436777711716349718	8.70000E-18	5.94858E-13	3.33067E-16
0.800	0.66403677026784896369	0.66403677026784896536	1.67000E-18	5.32796E-13	5.55112E-16
0.900	0.71629787019902442081	0.71629787019902442264	1.83000E-18	4.16112E-13	5.55112E-16
1.000	0.76159415595576488812	0.76159415595576488966	1.54000E-18	2.74558E-13	3.33067E-16

Table 4.4: Results for Problem 4.4

t	**Analytic Solution**	**Approximate Solution**	**ENNS**	**AES17**
0.100	-0.09966799462495581712	-0.09966799462495872645	2.90933E-15	1.14769E-14
0.200	-0.19737532022490400074	-0.19737532022490851498	4.51424E-15	6.71407E-14
0.300	-0.29131261245159090582	-0.29131261245159518500	4.27918E-15	1.83409E-13
0.400	-0.37994896225522488527	-0.37994896225522751844	2.63317E-15	3.38563E-13
0.500	-0.46211715726000975850	-0.46211715726001029934	5.40840E-16	4.86111E-13
0.600	-0.53704956699803528586	-0.53704956699803416361	1.12225E-15	5.79869E-13
0.700	-0.60436777711716349631	-0.60436777711716155189	1.94442E-15	5.94747E-13
0.800	-0.66403677026784896368	-0.66403677026784698481	1.97887E-15	5.32907E-13
0.900	-0.71629787019902442081	-0.71629787019902289368	1.52713E-15	4.16001E-13
1.000	-0.76159415595576488812	-0.76159415595576397424	9.13880E-16	2.74447E-13

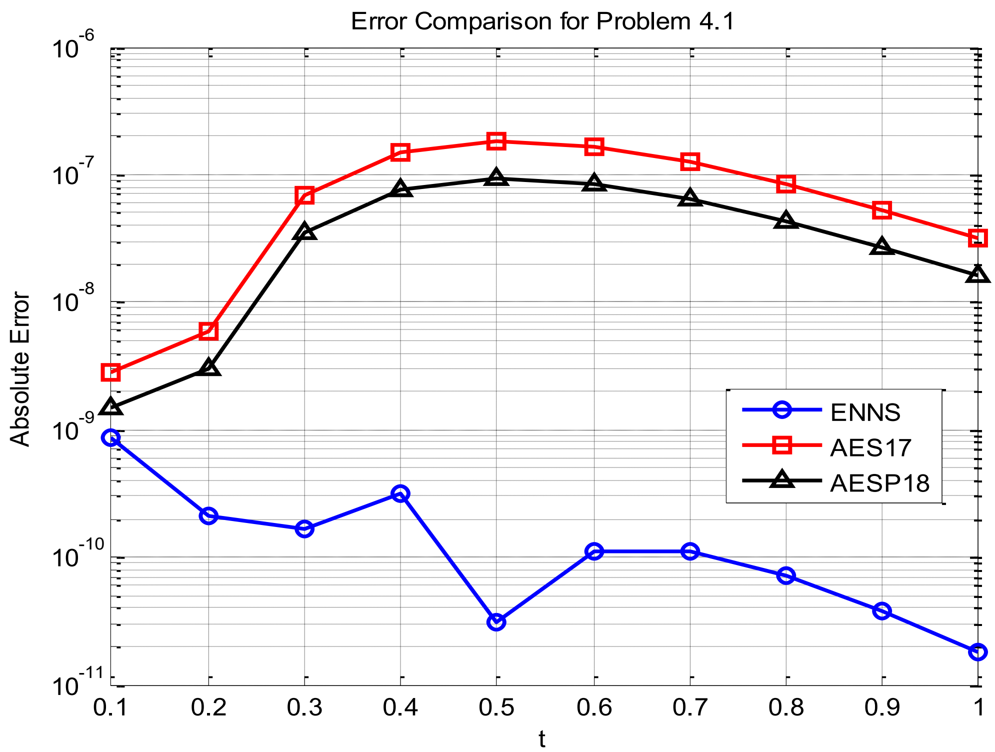

Figure 4.1: the Textual Curve of table 4.1

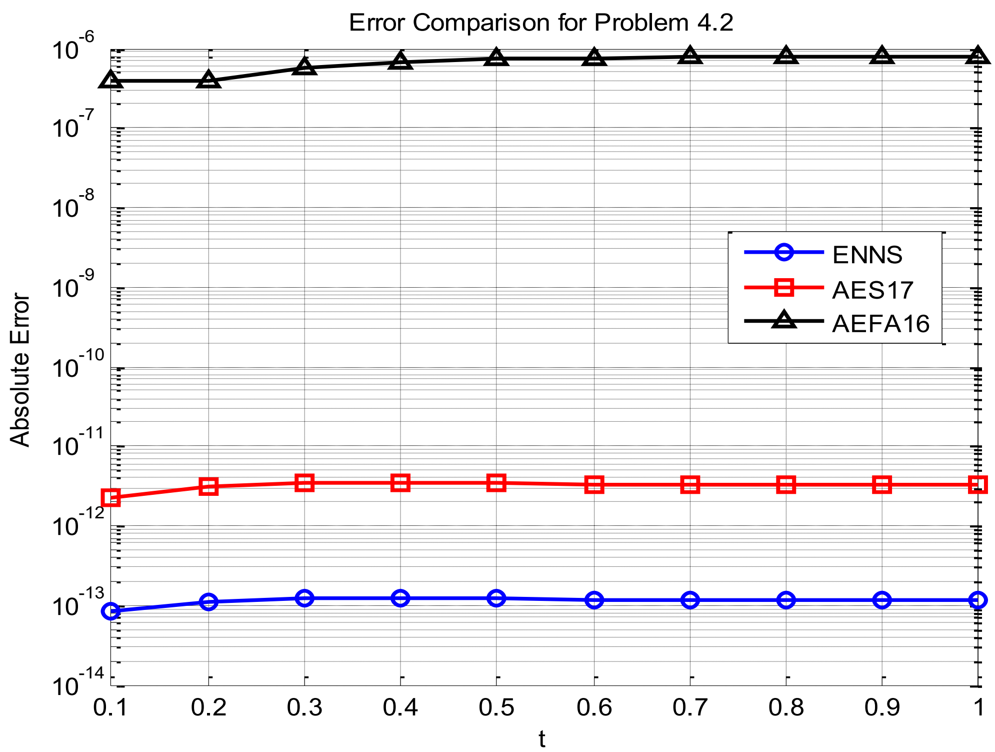

Figure 4.2: the Textual Curve of table 4.2

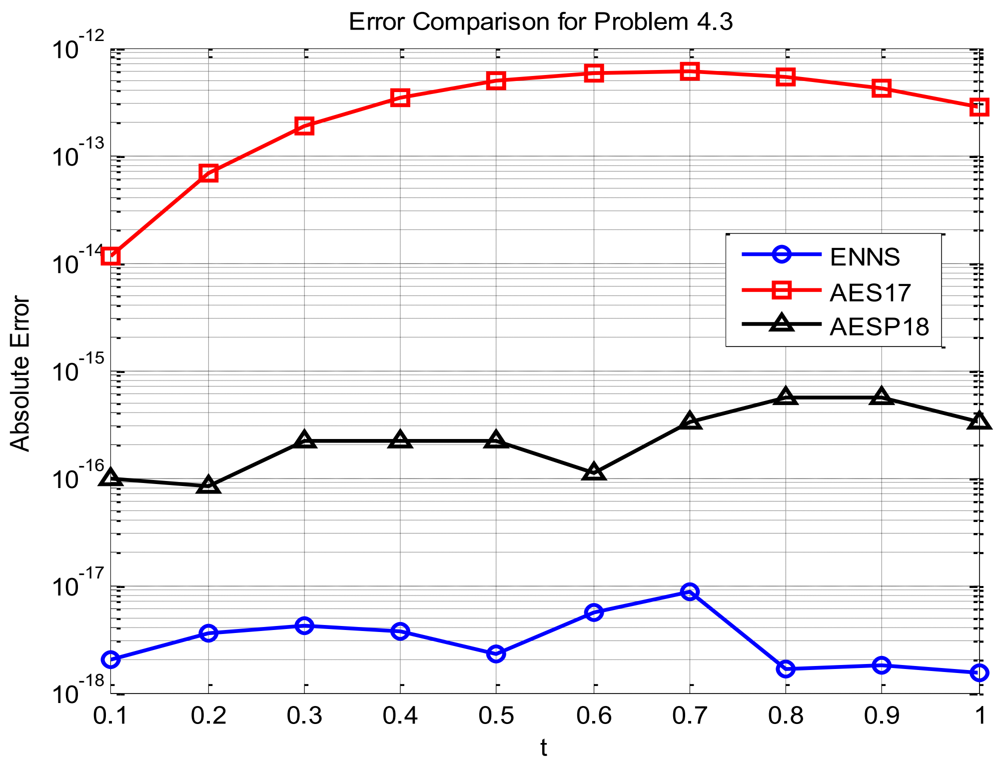

Figure 4.3: the Textual Curve of table 4.3

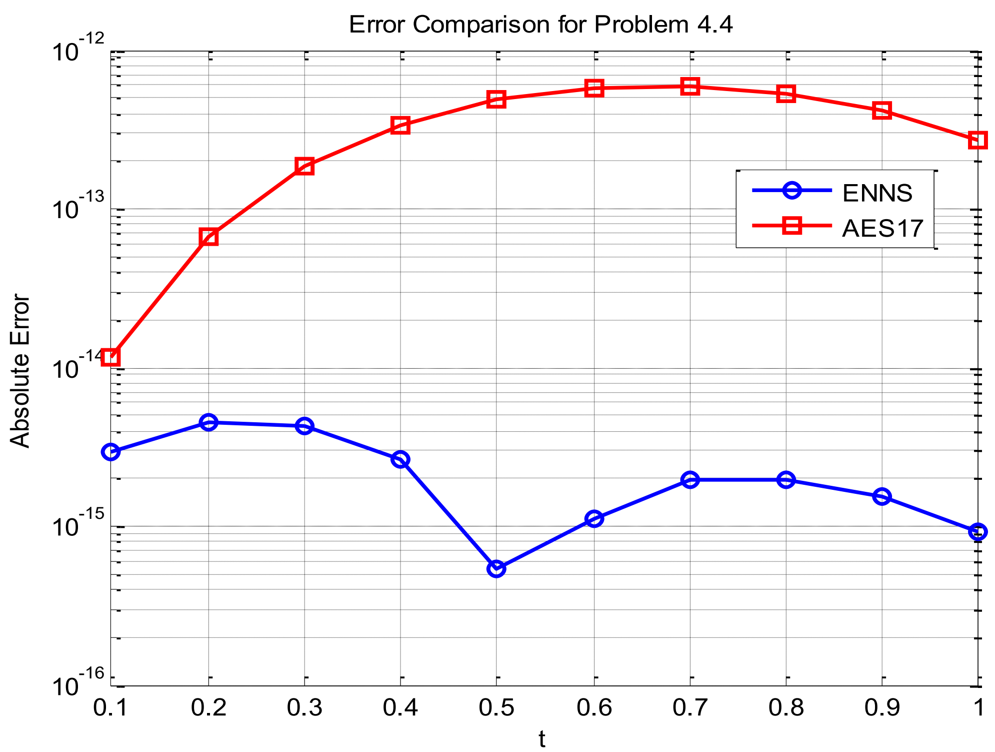

Figure 4.4: the Textual Curve of table 4.4

Discussion

In this part, a New Numerical Scheme (NNS) intending to solve Quadratic Riccati Differential Equations (QRDEs) is presented by using a power series basis function to approximate the exact solution within the one-step interval of integration. The interval is divided into appropriate subgrids, and the approximate solution is built by employing a mix of interpolation and collocation methods. Through interpolation, it is guaranteed that the proposed power series agrees with the solution at the chosen grid points, while collocation of the derivative form of the QRDE at several points turns the continuous problem into a system of nonlinear algebraic equations. By considering a power series of order eight, the method gets enough versatility to faithfully depict the behavior of the underlying differential equation. The explicit coefficients obtained by the Gauss elimination technique for the resulting nonlinear system are then substituted back into the series to derive both the continuous and discrete forms of the NNS appropriate for numerical implementation.

The accuracy, stability, and convergence of the suggested NNS are thoroughly examined. By Taylor series expansion, the associated linear operator verifies that the method uniformly reaches the eighth order of accuracy along with a corresponding error constant which means the rapid decrease of truncation error with the reduction of the step size. Zero stability is proved through the root condition as all the roots of the stability polynomial are inside the unit circle while consistency is obtained directly from the order of accuracy achieved. The combination of consistency and zero stability as per the Dahlquist convergence theorem speaks of the convergence of the method. Besides, the absolute stability analysis shows that the NNS is stable even when the step sizes are very different, indicating wide range of use for QRDEs, making the NNS a reliable and efficient method for numerical solution.

In this part of the text, the numerical application of the New Numerical Scheme (NNS) is shown quantitatively by checking its performance on four Quadratic Riccati Differential Equations (QRDEs) with exact solutions known. The problems chosen, indicated as Problems 4.1-4.4, are taken from the literature to offer a proper validation of the scheme. Each problem is a distinct type of QRDE, thus making sure the NNS capability is tested with different solution types from positive to negative ones. The NNS-generated approximate solutions are compared directly with the analytic solutions that correspond to them over the interval of integration, and the results are presented in Tables 4.1-4.4.

The outcomes for Problems 4.1 and 4.2 indicate that the approximate solutions derived from the NNS are almost identical to the exact analytic solutions at all the grid points considered. In the case of Problem 4.1, the numerical solutions not only follow the exact solution throughout the interval but also indicate the scheme's ability to capture the dynamics of the underlying Riccati equation with good accuracy. A more or less similar extent of agreement is noted in Problem 4.2, where the NNS is able to accurately mirror the exact solution’s smooth decay. The very small difference between the analytical and the approximate solutions attests to the accuracy and dependability of the technique proposed.

NNS shows a very good agreement even in the presence of non-linear growth and negative profiles in Problem 4.3 and Problem 4.4. The NNS numerical solutions have at least the same quality as the analytical ones over the whole interval in Problem 4.3. The same goes in Problem 4.4, where tracking of the exact solution was very accurate by the scheme despite the change in sign and increased nonlinearity of the problem. Thus, the NNS seems to be a method that is both robust and flexible with respect to different kinds of QRDEs.

The NNS performance is additionally depicted in Figures 4.1–4.4 through the visual curves of the analytical and approximate solutions for every single test case. It is very clear from every figure that the curves concerning the NNS solutions are nearly completely located on top of the exact solution curves, thus offering strong visual proof of the method's very high precision. The numerical curves' smoothness and stability can also be taken as an indication of the good numerical behavior throughout the whole integration interval. To sum up, the numerical tests prove that the New Numerical Scheme is of great accuracy, stability and efficiency when dealing with Quadratic Riccati Differential Equations, which in turn confirms its suitability for real-world applications.

Conclusion

The research work has presented a New Numerical Scheme (NNS) for the effective treatment of Quadratic Riccati Differential Equations (QRDEs). These equations are very common in control theory, nonlinear dynamics, and applied mathematics. The NNS is constructed by utilizing an eighth-order power series basis function along with interpolation and collocation methods, thereby providing an accurate approximation of the exact solution in every integration interval. The non-linear algebraic equations obtained are then efficiently solved by the Gauss elimination method. Theoretical analysis of the method verified its very high order of accuracy, zero stability, consistency, and convergence along with absolute stability. The four test QRDEs with known exact solutions of numerical applications demonstrated that the NNS replicates the analytic solutions very closely for different types of problems, including those with positive, negative, and nonlinear solution profiles. The method's accuracy and robustness were further evidenced by graphical illustrations showing almost perfect coincidence between the numerical and exact solutions.

In this study, a New Numerical Scheme (NNS) has been developed that can be regarded as a trustworthy, precise, and stable method for QRDEs' solution. Its very high order of accuracy and exceptional performance in different test cases validate that it is avant-garde among the already existing methods in terms of precision and numerical stability. The NNS has been confirmed as a durable method for use in various engineering, physics, and applied mathematics applications where QRDEs are frequently encountered. The method was proven through both theoretical analysis and numerical experiments. As a result, the NNS gives researchers and practitioners working with nonlinear differential equations a computational tool that not only is efficient but also provides accuracy and computational economy, in a nutshell, the NNS is a powerful and practical computational tool.

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