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A Continuous Class of A-Stable Block Generalized Backward Differentiation Formulae for Solving Stiff Problems of Ordinary Differential Equations

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Abstract

This paper presents the development and analysis of a continuous class of A-stable Block Generalized Backward Differentiation Formulae (BBDF) for the numerical solution of stiff ordinary differential equations (ODEs). The proposed methods, denoted as 4SCBGBDF and 4SHBGBDF, extend the conventional BBDF framework by incorporating continuous interpolation functions, which facilitate the generation of dense output without incurring additional computational cost. The block structure of these methods enables the simultaneous computation of multiple solution points within a single step, thereby enhancing computational efficiency and solution accuracy. A rigorous stability analysis confirms the A-stability of both methods, affirming their suitability for stiff initial value problems. Numerical experiments conducted on standard benchmark stiff problems validate the theoretical properties and demonstrate the superior performance of the proposed methods in terms of stability, accuracy, and computational efficiency. The results underscore the potential of continuous A-stable BBDF schemes as robust and reliable tools for solving stiff systems arising in scientific and engineering applications.

Keywords: A-Stability; Stiff Systems; Linear Multistep Methods; Block Off-Step Point; Backward Differentiation Formula; Ordinary Differential Equations

INTRODUCTION

Stiff ordinary differential equations (ODEs) are a class of differential equations where traditional numerical methods (such as explicit Runge-Kutta methods) fail or require extremely small step sizes to maintain stability. Stiffness arises in many scientific and engineering applications, including chemical kinetics, control systems, and circuit simulations.

To efficiently solve stiff ODEs, implicit methods such as the Backward Differentiation Formulae (BDF) are commonly used. BDF methods are known for their strong stability properties, making them suitable for stiff problems. However, traditional BDF methods are limited by their fixed step-size implementation and order constraints.

A class of numerical techniques used to resolve normal differential techniques (NDAs) that are thought to be both accurate and numerically stable is known as "A-stable block methods." These techniques intentionally maintain stability for stiff and real-life modeled issues, which are distinguished by rapid changes in the system's dynamics. Generalized backward differentiation formulae (GBDF) are one way to achieve A-stability in block approaches (Omar & Kuboye, 2015).

The backward differentiation formulae (BDF) remains a family of implicit numerical techniques used for solving routine differential equations (ODEs). These methods are mostly effective for solving stiff ODEs, which are problems that have solutions with speedily changing behavior on different time scales. BDF methods are multi-step methods based on backward differentiation formulae, which use weighted combinations of past derivatives to approximate the solution at the current time step. BDF methods are mostly suitable for stiff problems and modeled real-life problems because they can be considered to have high order accuracy while maintaining stability (Atsi & Kumleng 2020).

Recent advancements have led to block methods and generalized BDF (GBDF) approaches, which allow for variable step sizes and higher-order approximations while maintaining stability. The Continuous Block Generalized BDF (CBGBDF) is an extension that combines the benefits of block methods (simultaneous solution at multiple points) with the stability of BDF, offering improved accuracy and efficiency for stiff ODEs (Onsachi, et al., 2020).

Collocation methods are widely considered as a way of generating numerical solution to ordinary differential equation of the form:

$$y'(x) = f(x, y(x)),\ y(x_0) = y_0,\ x \in [x_0, x_N] \tag{1.1}$$

where $f\ :[x_0, x_N] \times R^n \to R^m,\quad y_0 \in R^n$ is continuous and differentiable. However, it is assumed that (1.1) satisfy the existence and uniqueness theorem within the interval of integration.

Stiff ODEs frequently appear in chemical reaction kinetics, where reactions occur at vastly different timescales (Gear, 1971). The proposed A-stable block generalized backward differentiation formulae (BDF) can efficiently handle such systems, ensuring numerical stability and accuracy when solving complex reaction networks (Hairer & Wanner, 1996). In the analysis of electrical circuits, particularly those involving parasitic capacitances and inductances, stiff systems emerge due to widely varying time constants. The A-stable property of these BDF methods makes them suitable for simulating such circuits without encountering instability (Ascher & Petzold, 1998). Stiff ODEs are common in control theory, especially in systems with fast and slow dynamics, such as robotic manipulators with high-frequency actuators. The block generalized BDF methods can provide stable and efficient solutions for real-time control applications. Climate models often involve stiff equations due to the coupling of slow (ocean dynamics) and fast (atmospheric turbulence) processes (Hindmarsh et al., 2005). The A-stability of these methods ensures reliable long-term simulations without artificial numerical damping. Modeling biological processes, such as enzyme kinetics or drug absorption, often leads to stiff ODEs. The proposed method can accurately capture the dynamics of such systems, making it useful in drug development and systems biology (Butcher, 2008). In structural dynamics, stiff systems arise when modeling structures with both rigid and flexible components (Brenan et al., 1996). The block BDF approach ensures stability in simulating high-frequency vibrations coupled with slow mechanical motions.

Motivated from the above literature, this study is aimed at developing both the conventional and hybrid method of four step block generalized backward differentiation formulae 4SCBGBDF and 4SHBGBDF respectively using polynomial basis function to solve stiff problems of first order ordinary differential equations ODEs. The method will be employed in solving stiff chemical reaction problems of the form (1.1). Several points have been examined for the selection of off-step points and hence it was concluded that selecting the points where the step is halved will lead to zero-stable formulae. Compared to the methods developed by Yakubu (2018), Yakubu and Markus (2016) Yakubu, Kwami and Ahmed (2012), the advantage of the proposed methods is that the solutions are approximated with off-step points concurrently and provide better accuracy.

METHODS

Four-Step Conventional Block Generalized Backward Differentiation Formulae (4SCBGBDF)

Consider k- step discrete formula

$$y(x_{n+k}) = \sum_{j=0}^{k-1} \alpha_j y_{n+j} + h \sum_{j=0}^{k-1} \beta_j f(\overline{x}_j, y(\overline{x})) \tag{2.1}$$

from equation (2.1). The continuous formulation of GBDF becomes

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + h\beta_v f_{n+v} \tag{2.2}$$

we consider $j = v$ in the form of

$$j = v \equiv \begin{cases} \frac{k-1}{2}, \text{ when k is odd} \\ \frac{k-2}{2}, \text{ when k is even} \end{cases} \tag{2.3}$$

to develop four step conventional method when $4 = 2,$ then $v = 1,$ according to equation (2.3). The continuous formulation of GBDF becomes

$$y(x) = \sum_{j=0}^{3} \alpha_j(x) y_{n+j} + h\beta_1(x) f_{n+1} \tag{2.4}$$

Expressing (2.4) in matrix form accordingly gives

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 \\ 0 & 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 \end{pmatrix} \tag{2.5}$$

Using MAPLE 18 software code or Gaussian elimination method, the continuous coefficients $\alpha_0(\xi), \alpha_1(\xi), \alpha_2(\xi), \alpha_3(\xi)$ and $\beta_1(\xi)$ derived and substituted into (2.1) giving the continuous formulation of GBDF for $k = 3$ (3SCGBDF).

$$\begin{pmatrix} y(\xi) = \alpha_0(\xi) y_n + \alpha_1(\xi) y_{n+1} \\ +\alpha_2(\xi) y_{n+2} + \alpha_3(\xi) y_{n+3} + h\beta_1(\xi) f_{n+1} \end{pmatrix} \tag{2.6}$$

where

$$\alpha_0(\xi) = \frac{1}{6}\xi^4 - \frac{7}{6}\xi^3 + \frac{17}{6}\xi^2 - \frac{17}{6}\xi + 1$$

$\alpha_1(\xi) = \frac{1}{4}\xi^4 - \xi^3 + \frac{1}{4}\xi^2 + \frac{3}{2}\xi$

$\alpha_2(\xi) = -\frac{1}{2}\xi^4 + \frac{5}{2}\xi^3 - \frac{7}{2}\xi^2 + \frac{3}{2}\xi$

$\alpha_3(\xi) = \frac{1}{12}\xi^4 - \frac{1}{3}\xi^3 + \frac{5}{12}\xi^2 - \frac{1}{6}\xi$

$\beta_1(\xi) = \frac{1}{2}\xi^4 - 3\xi^3 + \frac{11}{2}\xi^2 - 3\xi$

and $\xi = x - x_n, \quad 0 \le \xi \le 4h$

Evaluating (2.6) at $\xi = 4h$ and its first derivative at $\xi = 0, 2h, 3h, 4h.$ After collecting the terms and putting it in a block form we obtain four step conventional block method (4SCBM) in the form of

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{n-1} \\ y_{n-2} \\ y_{n-3} \\ y_n \end{pmatrix} + h\begin{pmatrix} 0 & 0 & 0 & \frac{3}{8} \\ 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{3}{8} \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} f_{n-1} \\ f_{n-2} \\ f_{n-3} \\ f_n \end{pmatrix}$$

$$+h\begin{pmatrix} \frac{19}{24} & -\frac{5}{24} & \frac{1}{24} & 0 \\ \frac{4}{3} & \frac{1}{3} & 0 & 0 \\ \frac{9}{8} & \frac{9}{8} & \frac{3}{8} & 0 \\ \frac{8}{3} & -\frac{4}{3} & \frac{8}{3} & 0 \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{pmatrix} \tag{2.7}$$

and the normalized discrete Scheme, called four steps conventional normalized discrete scheme (4SCNDS) as

$$y_{n+1} = y_n + h\left[\frac{3}{8}f_n + \frac{19}{24}f_{n+1} - \frac{5}{24}f_{n+2} + \frac{1}{24}f_{n+3}\right]$$

$$y_{n+2} = y_n + h\left[\frac{1}{3}f_n + \frac{4}{3}f_{n+1} + \frac{1}{3}f_{n+2}\right]$$

$$y_{n+3} = y_n + h\left[\frac{3}{8}f_n + \frac{9}{8}f_{n+1} + \frac{9}{8}f_{n+2} + \frac{3}{8}f_{n+3}\right]$$

$$y_{n+4} = y_n + h\left[\frac{8}{3}f_{n+1} - \frac{4}{3}f_{n+2} + \frac{8}{3}f_{n+3}\right] \tag{2.8}$$

Four-Step Block Hybrid Generalized Backward Differentiation Formulae (4SBHGBDF)

It follows from (2.3),

when $k = 4, v = 1,$ the continuous formulation of 4SBHGBDF (2.1) becomes

$$y(x) = \sum_{j=0}^{3} \alpha_j(x) y_{n+j} + h\beta_1(x) f_{n+1} \tag{2.9}$$

Expressing (2.9) accordingly gives

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 1 & x_{n+\frac{1}{2}} & x_{n+\frac{1}{2}}^2 & x_{n+\frac{1}{2}}^3 & x_{n+\frac{1}{2}}^4 & x_{n+\frac{1}{2}}^5 & x_{n+\frac{1}{2}}^6 & x_{n+\frac{1}{2}}^7 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\ 1 & x_{n+\frac{3}{2}} & x_{n+\frac{3}{2}}^2 & x_{n+\frac{3}{2}}^3 & x_{n+\frac{3}{2}}^4 & x_{n+\frac{3}{2}}^5 & x_{n+\frac{3}{2}}^6 & x_{n+\frac{3}{2}}^7 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 \\ 1 & x_{n+\frac{5}{2}} & x_{n+\frac{5}{2}}^2 & x_{n+\frac{5}{2}}^3 & x_{n+\frac{5}{2}}^4 & x_{n+\frac{5}{2}}^5 & x_{n+\frac{5}{2}}^6 & x_{n+\frac{5}{2}}^7 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 \\ 0 & 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 \end{pmatrix} \tag{3.0}$$

Using MAPLE 18 software code or Gaussian elimination method, the continuous coefficients $\alpha_0(\xi), \alpha_{\frac{1}{2}}(\xi), \alpha_1(\xi), \alpha_{\frac{3}{2}}(\xi), \alpha_2(\xi), \alpha_{\frac{5}{2}}(\xi), \alpha_3(\xi)$ and $\beta_1(\xi)$ are derived and substituted into (2.1) giving the continuous formulation of GBDF for $k = 4$ (4SHGBDF) as

$$\begin{pmatrix} y(\xi) = \alpha_0(\xi) y_n + \alpha_{\frac{1}{2}}(\xi) y_{n+\frac{1}{2}} \\ +\alpha_1(\xi) y_{n+1} + \alpha_{\frac{3}{2}}(\xi) y_{n+\frac{3}{2}} + \alpha_2(\xi) y_{n+2} \\ +\alpha_{n+\frac{5}{2}}(\xi) y_{n+\frac{5}{2}} + \alpha_3(\xi) y_{n+3} + h\beta_1(\xi) f_{n+1} \end{pmatrix} \tag{3.1}$$

where

$$\alpha_0(\xi) = -\frac{1}{90}(8\xi^7 - 92\xi^6 + 434\xi^5 - 1085\xi^4 + 1547\xi^3 - 1253\xi^2 + 531\xi - 90)$$

$$\alpha_{\frac{1}{2}}(\xi) = \frac{4}{15}(4\xi^7 - 44\xi^6 + 195\xi^5 - 445\xi^4 + 551\xi^3 - 351\xi^2 + 90\xi)$$

$$\alpha_1(\xi) = \frac{1}{36}(56\xi^7 - 540\xi^6 + 1994\xi^5 - 3501\xi^4 + 2918\xi^3 - 981\xi^2 + 90\xi)$$

$$\alpha_{\frac{3}{2}}(\xi) = -\frac{8}{9}(4\xi^7 - 40\xi^6 + 157\xi^5 - 307\xi^4 + 313\xi^3 - 157\xi^2 + 30\xi)$$

$$\alpha_2(\xi) = \frac{1}{6}(8\xi^7 - 76\xi^6 + 282\xi^5 - 521\xi^4 + 505\xi^3 - 243\xi^2 + 45\xi)$$

$$\alpha_{n+\frac{5}{2}}(\xi) = -\frac{4}{45}(4\xi^7 - 36\xi^6 + 127\xi^5 - 225\xi^4 + 211\xi^3 - 99\xi^2 + 18\xi)$$

$$\alpha_3(\xi) = \frac{1}{180}(8\xi^7 - 68\xi^6 + 230\xi^5 - 395\xi^4 + 362\xi^3 - 167\xi^2 + 30\xi)$$

$$\beta_1(\xi) = \frac{1}{6}(8\xi^7 - 84\xi^6 + 350\xi^5 - 735\xi^4 + 812\xi^3 - 441\xi^2 + 90\xi)$$

and $\xi = x - x_n, \quad 0 \le \xi \le 4h$

Evaluating (3.1) at $\xi = \frac{7}{2}h, 4h$ and its first derivative at $\xi = 0\left(\frac{1}{2}\right)3h$. After collecting the terms and putting it in a block form we obtain four step hybrid block method (4SHBM) in the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \\ y_{n+\frac{5}{2}} \\ y_{n+3} \\ y_{n+\frac{7}{2}} \\ y_{n+4} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-\frac{1}{2}} \\ y_{n-1} \\ y_{n-\frac{3}{2}} \\ y_{n-2} \\ y_{n-\frac{5}{2}} \\ y_{n-3} \\ y_{n-\frac{7}{2}} \\ y_n \end{pmatrix}$$

$$+h \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{19087}{120960} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1139}{7560} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{137}{896} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{143}{945} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3715}{24192} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{41}{280} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5257}{17280} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{368}{189} \end{pmatrix} \begin{pmatrix} f_{n-\frac{1}{2}} \\ f_{n-1} \\ f_{n-\frac{3}{2}} \\ f_{n-2} \\ f_{n-\frac{5}{2}} \\ f_{n-3} \\ f_{n-\frac{7}{2}} \\ f_n \end{pmatrix} +$$

$$h\begin{pmatrix} \frac{2713}{5040} & -\frac{15487}{40320} & \frac{293}{945} & -\frac{6737}{40320} & \frac{263}{5040} & -\frac{863}{120960} \\ \frac{47}{63} & \frac{11}{2520} & \frac{166}{945} & -\frac{269}{2520} & \frac{11}{315} & -\frac{37}{7560} \\ \frac{81}{112} & \frac{1161}{4480} & \frac{17}{35} & -\frac{729}{4480} & \frac{27}{560} & -\frac{29}{4480} \\ \frac{232}{315} & \frac{64}{315} & \frac{752}{945} & \frac{29}{315} & \frac{8}{315} & -\frac{4}{945} \\ \frac{725}{1008} & \frac{2125}{8064} & \frac{125}{189} & \frac{3875}{8064} & \frac{235}{1008} & -\frac{275}{24192} \\ \frac{27}{35} & \frac{27}{280} & \frac{34}{35} & \frac{27}{280} & \frac{27}{35} & \frac{41}{280} \\ -\frac{49}{144} & \frac{19943}{5760} & -\frac{637}{135} & \frac{34153}{5760} & -\frac{2107}{720} & \frac{30919}{17280} \\ -\frac{736}{63} & \frac{11608}{315} & -\frac{55616}{945} & \frac{18188}{315} & -\frac{9952}{315} & \frac{9064}{945} \end{pmatrix}\begin{pmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \\ f_{n+\frac{7}{2}} \\ f_{n+4} \end{pmatrix} \qquad (3.2)$$

and the normalized discrete Schemes, four step hybrid normalized discrete scheme (4SHNDS) as

$$y_{n+\frac{1}{2}} = y_n + h\begin{bmatrix} \frac{19087}{120960} f_n + \frac{2713}{5040} f_{n+\frac{1}{2}} - \frac{15487}{40320} f_{n+1} + \frac{293}{945} f_{n+\frac{3}{2}} \\ -\frac{6737}{40320} f_{n+2} + \frac{263}{5040} f_{n+\frac{5}{2}} - \frac{863}{120960} f_{n+3} \end{bmatrix}$$

$$y_{n+1} = y_n + h\begin{bmatrix} \frac{1139}{7560} f_n + \frac{47}{63} f_{n+\frac{1}{2}} + \frac{11}{2520} f_{n+1} + \frac{166}{945} f_{n+\frac{3}{2}} \\ -\frac{269}{2520} f_{n+2} + \frac{11}{315} f_{n+\frac{5}{2}} - \frac{37}{7560} f_{n+3} \end{bmatrix}$$

$$y_{n+\frac{3}{2}} = y_n + h\begin{bmatrix} \frac{137}{896} f_n + \frac{81}{112} f_{n+\frac{1}{2}} + \frac{1161}{4480} f_{n+1} + \frac{17}{35} f_{n+\frac{3}{2}} \\ -\frac{729}{4480} f_{n+2} + \frac{27}{560} f_{n+\frac{5}{2}} - \frac{29}{4480} f_{n+3} \end{bmatrix}$$

$$y_{n+2} = y_n + h\begin{bmatrix} \frac{143}{945} f_n + \frac{232}{315} f_{n+\frac{1}{2}} + \frac{64}{315} f_{n+1} + \frac{752}{945} f_{n+\frac{3}{2}} \\ +\frac{29}{315} f_{n+2} + \frac{8}{315} f_{n+\frac{5}{2}} - \frac{4}{945} f_{n+3} \end{bmatrix}$$

$$y_{n+\frac{5}{2}} = y_n + h\begin{bmatrix} \frac{3715}{24192} f_n + \frac{725}{1008} f_{n+\frac{1}{2}} + \frac{2125}{8064} f_{n+1} + \frac{125}{189} f_{n+\frac{3}{2}} \\ +\frac{3875}{8064} f_{n+2} + \frac{235}{1008} f_{n+\frac{5}{2}} - \frac{275}{24192} f_{n+3} \end{bmatrix}$$

$$y_{n+3} = y_n + h\begin{bmatrix} \frac{41}{280} f_n + \frac{27}{35} f_{n+\frac{1}{2}} + \frac{27}{280} f_{n+1} + \frac{34}{35} f_{n+\frac{3}{2}} \\ +\frac{27}{280} f_{n+2} + \frac{27}{35} f_{n+\frac{5}{2}} + \frac{41}{280} f_{n+3} \end{bmatrix}$$

$$y_{n+\frac{7}{2}} = y_n + h\begin{bmatrix}\frac{5257}{17280}f_n - \frac{49}{144}f_{n+\frac{1}{2}} + \frac{19943}{5760}f_{n+1} - \frac{637}{135}f_{n+\frac{3}{2}} \\ +\frac{34153}{5760}f_{n+2} - \frac{2107}{720}f_{n+\frac{5}{2}} + \frac{30919}{17280}f_{n+3}\end{bmatrix}$$

$$y_{n+4} = y_n + h\begin{bmatrix}\frac{368}{189}f_n - \frac{736}{63}f_{n+\frac{1}{2}} + \frac{11608}{315}f_{n+1} - \frac{55616}{945}f_{n\frac{3}{2}} \\ +\frac{18188}{315}f_{n+2} - \frac{9952}{315}f_{n+\frac{5}{2}} + \frac{9064}{945}f_{n+3}\end{bmatrix} \tag{3.3}$$

Stability Properties of the Methods

The following section comprises the analysis of basic properties of the proposed methods. These properties include order, consistency, zero-stability and convergence. The region of absolute stability of the method shall also be determined.

Order and Error Constant Hybrid Methods

Definition 3.1 (*Order and Error Constant*). The LMM (2.1) associated with the linear difference operator is said to be of order p if $C_1 = C_2 = \cdots = C_p = 0$ and $C_{p+1} \neq 0$. The term $C_{p+1} \neq 0$ is called the error constant of the method. By using tailor's series expansion on (2.8) and (3.3) the order and error constant of the block generalized backward differentiation formulae (BGBDF) are presented in table (1)

Table 1: Order and Error Constants of the new methods

Methods	Order	Error Constant
method (4SCBGBDF)	$(i)\ y_{n+1},\ p=4$	$-\frac{19}{720}$
	$(ii)\ y_{n+2},\ p=4$	$-\frac{1}{90}$
	$(iii)\ y_{n+3},\ p=4$	$-\frac{3}{80}$
	$(iv)\ y_{n+4},\ p=4$	$\frac{14}{45}$
method (4SHBGBDF)	$(i)\ y_{n+\frac{1}{2}},\ p=7$	$\frac{275}{6193152}$
	$(ii)\ y_{n+1},\ p=7$	$\frac{1}{30240}$
	$(iii)\ y_{n+\frac{3}{2}},\ p=7$	$\frac{9}{229376}$
	$(iv)\ y_{n+2},\ p=7$	$\frac{1}{30240}$
	$(v)\ y_{n+\frac{5}{2}},\ p=7$	$\frac{275}{6193152}$
	$(vi)\ y_{n+3},\ p=8$	$-\frac{9}{716800}$
	$(vii)\ y_{n+\frac{7}{2}},\ p=7$	$\frac{5257}{4423680}$
	$(viii)\ y_{n+4},\ p=7$	$\frac{23}{1512}$

Therefore, this can be demonstrated that the proposed methods are of order four and seven.

Consistency of the Methods

Definition 3.2 (*Consistency*)

The LMM (4SCBGBDF) and (4SHBGBDF) are said to be consistent if it is of order $p \geq 1$. It is therefore clear that they are consistent since it has order $p = 4$ and $p = 7$ respectively which satisfies Definition 3.2

Zero Stability Analysis of the Hybrid Methods

Definition 3.3 (*Zero-stable*)

If no root of the characteristic polynomial has a modulus greater than one and every root with modulus one is simple, then LMM (2.8) and (3.3) are said to be zero-stable.

$$\lambda\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}-\begin{pmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{pmatrix}=\begin{pmatrix}\lambda&0&0&-1\\0&\lambda&0&-1\\0&0&\lambda&-1\\0&0&0&\lambda-1\end{pmatrix},$$

determinant: $\lambda^4-\lambda^3=\lambda^3(\lambda-1)=0,$ Now, $\rho(\lambda)=0 \Leftrightarrow \lambda=0,0,0,1.$

$$\lambda\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{pmatrix}-\begin{pmatrix}0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\end{pmatrix}=$$

$$\begin{bmatrix}\lambda&0&0&0&0&0&0&-1\\0&\lambda&0&0&0&0&0&-1\\0&0&\lambda&0&0&0&0&-1\\0&0&0&\lambda&0&0&0&-1\\0&0&0&0&\lambda&0&0&-1\\0&0&0&0&0&\lambda&0&-1\\0&0&0&0&0&0&\lambda&-1\\0&0&0&0&0&0&0&\lambda-1\end{bmatrix},$$

determinant: $\lambda^8-\lambda^7=\lambda^7(\lambda-1)=0,$ Now, $\rho(\lambda)=0\Leftrightarrow\lambda=0,0,0,0,0,0,0,1.$

Since all the roots lie within $\lambda\leq 1$; hence, it is concluded that the methods 4SCBGBDF and 4SHBGBDF are zero-stable.

Convergence of the Methods

Definition 3.4 (*Convergence*): Consistency and zero-stability are required conditions for the LMM to be convergent. Therefore, we conclude that the newly derived 4SCBGBDF and 4SHBGBDF are convergent since it is consistent and zero-stable.

Stability Region of the Methods

Definition 3.5 (*A-stable*); The term "A-stable" refers to a method that is stable for all λ in the left-half plane $(Re(z)\leq 0)$. The stability region for A-stable method covers up the complete negative left-half plane.

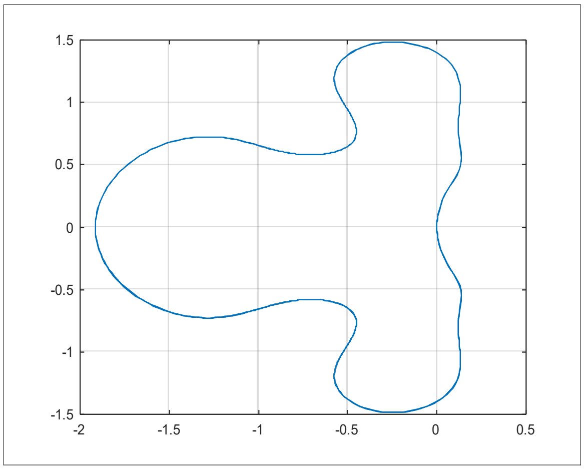

Figure 4.6: Region of Absolute Stability (RAS) of 4SCBGBDF

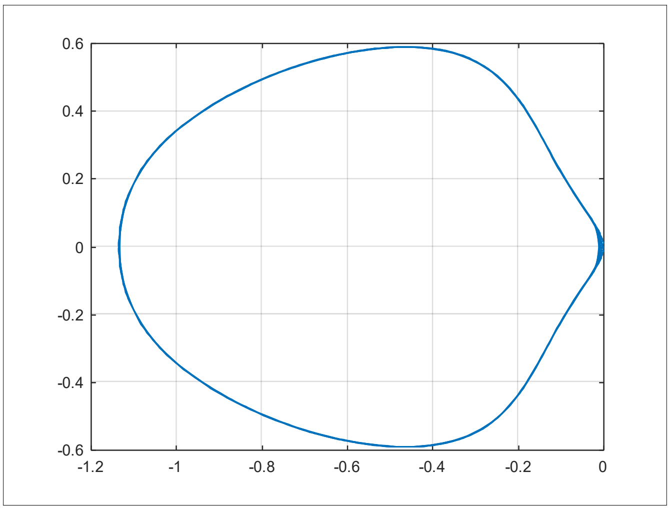

Figure 4.7: Region of Absolute Stability (RAS) of 4SHBGBDF

Definition: A numerical method is said to be A-stable if the whole of the left-half plane $\{Z \ : \ Re(Z) \leq 0\}$ is contained in the region. $\{Z: Re(Z) \leq 1\}$. Where $R(Z)$ is called the stability polynomial of the method. Lambert, (1973). The region of absolute stability of the newly constructed block methods are plotted with the help of MATLAB $R2015a$. This absolute stability regions are A-Stable since it consists of the set of points in the complex plane outside the enclosed figure

RESULTS

To test the reliability of the proposed 4SHBGBDF and 4SHBGBDF methods, some numerical results are obtained by applying the methods on some well-known stiff problems. Comparison is done with other multistep methods. The 4SHBGBDF and 4SHBGBDF methods has been coded and implemented with MATLAB $R2015a.$

Problem 1:

In this example we consider the stiff system with eigenvalues lying close to the imaginary axis, $-1 \pm 15i.$ This system is known to be troublesome stiff problem, where some stiff integrators were known to perform inefficiently.

$$y_1'(x) = -y_1(x) - 15y_2(x) + 15\,exp(-x), \quad y_1(0) = 1$$

$$y_2'(x) = 15y_1(x) - y_2(x) - 15\,exp(-x), \quad y_2(0) = 1.$$

The exact solution is,

$$y_1(x) = exp(-x)$$

$$y_2(x) = exp(-x)$$

We solve the system in the range of $[0,50]$ with $h = 0.1$ and the results obtained are presented in the table below. In this example we compare the results from one of the new methods of order six with another method from the literature side by side in table 4.1.

Table 4.1: Numerical Results and Comparison of Problem 1

x	y_i	Yakubu & Markus (2016)	Yakubu (2018)	4SCBGBDF	4SHBGBDF
	y_1	$6.2281E-02$	$6.4365E-04$	$5.59E-03$	$5.82E-05$
5	y_2	$3.0221E-02$	$2.5890E-04$	$7.50E-04$	$3.21E-05$
	y_1	$7.1266E-04$	$4.0701E-05$	$4.07E-06$	$3.41E-07$
50	y_2	$2.5951E-04$	$1.5333E-05$	$2.36E-06$	$2.44E-07$
	y_1	$3.1713E-08$	$1.8675E-09$	$6.55E-10$	$7.75E-11$
150	y_2	$1.1550E-08$	$1.3174E-09$	$4.42E-10$	$3.16E-12$
	y_1	$1.4112E-12$	$7.1404E-14$	$4.69E-15$	$7.21E-16$
250	y_2	$5.1387E-13$	$8.4689E-14$	$2.09E-15$	$2.28E-16$
	y_1	$1.8641E-23$	$7.0798E-26$	$6.80E-26$	$9.95E-27$
500	y_2	$6.7879E-24$	$1.3457E-24$	$9.56E-25$	$1.22E-26$

Problem 2

Consider the system of standard test equation with mildly stiff linear problem

$$y' = -8y_1 + 7y_2, \quad y_1(0) = 1$$

$$y' = 42y_1 - 43y_2, \quad y_2(0) = 8$$

The coefficient matrix of this problem has two eigenvalues, $\lambda_1 = -1$ and $\lambda_2 = -50$. The stiffness ratio is $R = 50$ with the exact solution

$$y_1(x) = 2\,exp(-x) - exp(-50x)$$

$y_2(x) = 2\,exp(-x) + 6\,exp(-50x)$ Source: (Yakubu, kwami and Ahmed 2012).

In this example we compare the results of our new methods 4SCBGBDF and 4SHBGBDF of order four and seven respectively with that of Yakubu *et al* (2012) of order eight in table 4.2.

Table 4.2: Numerical Results and Comparison of Problem 2

Mesh Values	Yakubu *et al* (2012)	4SCBGBDF	4SHBGBDF
10.0	1.04×10^{-2}	1.387798311148440e-05	4.917941269889188e-07
20.0	3.81×10^{-3}	1.636712378741021e-06	1.480423995515339e-08
30.0	1.34×10^{-5}	7.593155944870915e-08	6.313209034560852e-10
40.0	4.74×10^{-6}	5.069377561242702e-09	8.867922145795048e-12
50.0	1.67×10^{-8}	5.680810096148677e-11	2.591614855474534e-13
60.0	5.90×10^{-9}	5.754250179136318e-12	7.826764083023892e-15
70.0	2.08×10^{-11}	2.519492915926752e-14	2.376015958461080e-16
80.0	7.33×10^{-12}	3.130259490368326e-15	9.708524201978234e-18
90.0	2.58×10^{-14}	9.797632851266029e-17	9.748836597099413e-19
100.0	9.12×10^{-15}	1.077226981663674e-18	2.951134237877901e-21

DISCUSSION

Result of problem 1 in table 4.1 was picked from Yakubu (2018) work, shows that all our new methods perform better than Yakubu (2018) as the performance is displayed in the table. For 4SCBGBDF and 4SHBGBDF though same number step but the 4SHBGBDF perform better than 4SCBGBDF maybe because of the evaluations at intermediate points that enhance the accuracy. It also follows in problem 2, table 4.2, the problem was picked from Yakubu *et al* (2012) when compare with our new methods 4SCBGDBF and 4SHBGBDF outperform their own in terms of accuracy and error.

CONCLUSION

This study has successfully developed a continuous class of A-stable Block Generalized Backward Differentiation Formulae (BBDF) tailored for the numerical solution of stiff ordinary differential equations (ODEs). The method was successfully implemented using MATLAB R2015a software, stiff ordinary differential equations and methods was proven to be consistent, zero stable and convergent. The region of absolute stability of the method shows that is A-stable. The methods also demonstrated both theoretical and computational efficiency, exhibiting desirable properties such as A-stability, high order of accuracy, and suitability for stiff systems commonly encountered in chemical kinetics, control theory, and engineering applications. The continuous formulation enhances the flexibility of the methods, allowing for improved interpolation and dense output capabilities. Overall, the proposed class of methods contributes significantly to the advancement of stable and efficient numerical integrators for stiff ODE problems, providing a reliable tool for scientists and engineers in practical computation.

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