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Trigonometric-Fitted One-Step 3 Points Hybrid Block Method for the Solution of Stiff and Oscillating Problems

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Abstract

This paper introduces a novel Trigonometric-Fitted One-Step 3 Points Hybrid Block Method tailored for addressing the complexities associated with stiff and oscillating differential equations.The continuous hybrid technique was created using the interpolation method and the collocation of the trigonometrical function as the basis function. It was then evaluated at non-interpolating points by inculcating the transformation method to produce a continuous block method. When the continuous block was assessed at each stage, the discrete block approach was regained. Upon investigation, the fundamental characteristics of the techniques were discovered to be zero-stable, consistent, and convergent. The new method is used to solve a few stiff and oscillatory ordinary differential equation problems. Comparisons of numerical results of the derived methods, it was found that our approach provides a better approximation than the existing method cited in the reference.

AMS subject classification: 65L05, 65L06, 65L20

Keywords: One-step, Hybrid Point, Transformation, Trigometrically Fitted

Introduction

In this paper, we present a detailed exposition of the Trigonometric-Fitted One-Step 3 Points Hybrid Block Method, elucidating its formulation and elucidating the underlying principles by considering an approximate solution of second order ordinary differential equations using the one-step three off grid point hybrid approach of the type

$$y'' = f\left(x, y, y'\right), \quad y\left(t_0\right) = y_0, \quad y'\left(t_0\right) = y'_0, \qquad (1)$$

Numerous methods can be used to determine the known frequency of oscillation in the analytical solutions. Due to its numerous applications in a wide range of fields, such as theoretical physics and oscillatory motion, theoretical chemistry, classical mechanics, fluid dynamics, quantum mechanics, modeling scientific and engineering, celestial mechanics, and so forth, equation (1) is of particular interest to researchers. It may be difficult to solve several of these issues analytically, thus developing numerical techniques is necessary to get approximations of the solutions.

Numerous writers have presented numerous numerical algorithms in their works that precisely integrate a set of linearly independent variables for the solution of (1).[1] suggests a set of k-step block falkner methods that are trigonometrically fitted for solving equation (1); [2] suggests a backward differentiation formula that is trigonometrically fitted in blocks; and [3] suggests a two-step trigonometrically fitted method.

Among the scholars who have recently embraced the trigonometrically fitted approach in lieu of the direct method for approximating (1) are [4, 5, 6, 7, 8, 9, 10].

In this paper, we present a detailed exposition of the Trigonometric-Fitted One-Step 3 Points Hybrid Block Method, elucidating its formulation and elucidating the underlying principles. We showcase the method's capabilities through numerical experiments and comparisons with existing techniques, demonstrating its efficacy in solving stiff and oscillating problems. The structure of the paper is as follows: Section 2 covers the materials and techniques used in the method's development. In Section 3, the method's basis properties are analyzed, numerical experiments are conducted to test the developed method's efficiency on a few numerical examples, and the findings are discussed. Finally, we wrapped up in section 4.

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Derivation of the Method

The continuous representation of the one step trigonometric function as the approximate solution shall be derive to generate the main method which we shall set up to obtain the block method. We consider a trigonometric approximate solution of the form

$$\tau(\mathrm{t})=\sum_{j=0}^{q+p-1}\psi_j(\sin x+\cos x) \tag{2}$$

Equation (2) is obtained by considering the trigonometric function as approximate solution and

$p=5$ and $u=2$ are the numbers of points of collocation and interpolation, the second derivative of (2) gives

$$\tau''(x)=h^2\left(\sum_{j=2}^{q+p-1}\varphi_j(-\sin x-\cos x)\right)=f\left(x,\tau,\tau'\right) \quad \ldots \tag{3}$$

The continuouos approximation is then constructed by imposing two conditions which are

$$\left.\begin{aligned}\tau_{n+j}&=y\left(x_{n+j}\right),\quad j=0,\frac{1}{4}\\ \tau''\left(x_{n+j}\right)&=f_{n+j}\end{aligned}\right\} \tag{4}$$

Collocating (3) at all points and interpolating (2) at $u=0,\ \frac{1}{4}$ result to the system of non linear equation of the form

$$XA=U \tag{5}$$

The system of (5) is solve to obtain the unknown parameter φ_j's, $j=0(1)6$. By the substitutions of the values of φ_j's obtained into equation (2) and using transformation from [12] gives

$$\varphi_0 = \tau_n$$
$$\varphi_1 = \frac{4}{h}\tau_{n+\frac{1}{4}} - \frac{4}{h}\tau_n + \frac{1}{5760}h\left(1073 f_n + 21 f_{n+1} + 282 f_{n+\frac{1}{2}} - 540 f_{n+\frac{1}{4}} - 116 f_{n+\frac{3}{4}}\right)$$
$$\varphi_2 = f_n$$
$$\varphi_3 = \frac{1}{3}\frac{25 f_n + 3 f_{n+1} + 36 f_{n+\frac{1}{2}} - 48 f_{n+\frac{1}{4}} - 16 f_{n+\frac{3}{4}}}{h}$$
$$\varphi_4 = \frac{4}{3}\frac{35 f_n + 114 f_{n+\frac{1}{2}} - 104 f_{n+\frac{1}{4}} - 56 f_{n+\frac{3}{4}}}{h^2}$$
$$\varphi_5 = -32\frac{5 f_n + 3 f_{n+1} + 24 f_{n+\frac{1}{2}} - 18 f_{n+\frac{1}{4}} - 14 f_{n+\frac{3}{4}}}{h^3}$$
$$\varphi_6 = -256\frac{f_n + f_{n+1} + 16 f_{n+\frac{1}{2}} - 4 f_{n+\frac{1}{4}} - 4 f_{n+\frac{3}{4}}}{h^4} \qquad (6)$$

substituting (6) in (3) gives a continuous hybrid linear multistep method of the form

$$\tau(t) = \varphi_0(t)\tau_n + \varphi_{\frac{1}{4}}(t)\tau_{n+\frac{1}{4}} + h^2\left[\sum_{j=0}^{1}\psi_j(t)g_{n+j} + \sum_{i=\frac{1}{4}}^{\frac{3}{4}}\psi_i(t)g_{n+i}\right] \qquad (7)$$

We then impose (4) on $\tau(t)$ in (7) and the coefficients $\tau_{n+j}, i=0, \frac{1}{4}$ and $f_{n+j}, j=0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

Where $t = \frac{x - x_{n+2}}{h}, \frac{dt}{dx} = \frac{1}{h}$

$$\varphi_0 = -1$$
$$\varphi_{\frac{1}{4}} = -h\xi$$
$$\psi_0 = \frac{-1}{60}h^2\xi^3 \frac{3\xi^3-2\xi^4-5q\xi^2+3p\xi^3-5r\xi^2+3r\xi^3-5s\xi^2+3s\xi^3+10qr\xi+10qs\xi+10rs\xi-5qr\xi^2-5qs\xi^2-5rs\xi^2-30qrs+10qrs\xi}{qrs}$$
$$\psi_{\frac{1}{4}} = \frac{-1}{60}h^2\xi^3 \frac{5q\xi+5r\xi+3\xi^2+2\xi^3-3p\xi^2-3r\xi^2-10qr+5qr\xi}{s(s-1)(r-s)(q-s)}$$
$$\psi_{\frac{1}{2}} = \frac{-1}{60}h^2\xi^3 \frac{5q\xi+5s\xi-3\xi^2+2\xi^3-3q\xi^2-3s\xi^2-10qs+5qs\xi}{r(r-1)(r-s)(q-r)}$$
$$\psi_{\frac{3}{4}} = \frac{-1}{60}h^2\xi^3 \frac{5r\xi+5s\xi-3\xi^2+2\xi^3-3r\xi^2-3s\xi^2-10rs+5qr\xi}{q(q-1)(q-s)(q-r)}$$
$$\psi_1 = \frac{-1}{60}h^2\xi^3 \frac{-2\xi^3+3q\xi^2+3r\xi^2+3s\xi^2-5qr\xi-5qs\xi-5rs\xi+10qsr}{(r-1)(s-1)(q-1)}$$

differentiating of (7) once gives

$$\tau'(t)=\varphi'_0(t)\tau_n+\varphi'_{\frac{1}{4}}(t)\tau_{n+\frac{1}{4}}+h^2\left[\sum_{j=0}^{1}\psi'_j(t)g_{n+j}+\sum_{i=\frac{1}{4}}^{\frac{3}{4}}\psi'_i(t)g_{n+i}\right] \tag{8}$$

eveluating (7) and (8) at all points and simplifying gives the discrete hybrid block method of the form

$$A^0\tau_m^i=\sum_{i=0}^{1}\mathrm{h}^i e_i\tau_n^i+h^2 b_i f\left(\tau_n\right)+h^2 c_i f\left(\tau_m\right) \tag{9}$$

Where $A^{(0)}=4\times 4 \quad identity\ \ matrix$

$$\tau_m^i=\begin{bmatrix}\tau_{n+\frac{1}{4}} & \tau_{n+\frac{1}{2}} & \tau_{n+\frac{3}{4}} & \tau_{n+1}\end{bmatrix}^T, \quad \tau_n^i=\begin{bmatrix}\tau_{n-\frac{1}{2}} & \tau_{n-\frac{3}{4}} & \tau_{n-1} & \tau_n\end{bmatrix}^T$$

$$f\left(\tau_m^i\right)=\begin{bmatrix}f_{n+\frac{1}{4}} & f_{n+\frac{1}{2}} & f_{n+\frac{3}{4}} & f_{n+1}\end{bmatrix}^T, \quad f\left(\tau_n^i\right)=\begin{bmatrix}f_{n-\frac{1}{2}} & f_{n-\frac{3}{4}} & f_{n-1} & f_n\end{bmatrix}^T$$

$when\ i=0,1\ and\ \ r=\frac{1}{4},\ s=\frac{1}{2},\ q=\frac{3}{4}$

We obtain the following discrete scheme

when $i=0$

$$e_0=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix},\ e_1=\begin{bmatrix}0&0&0&\frac{1}{4}\\0&0&0&\frac{1}{2}\\0&0&0&\frac{3}{4}\\0&0&0&1\end{bmatrix},\ b_0=\begin{bmatrix}0&0&0&\frac{367}{23040}\\0&0&0&\frac{53}{1440}\\0&0&0&\frac{147}{2560}\\0&0&0&\frac{7}{90}\end{bmatrix},\ c_0=\begin{bmatrix}\frac{3}{128}&\frac{-47}{3840}&\frac{29}{5760}&\frac{-7}{7680}\\\frac{1}{10}&\frac{-1}{48}&\frac{1}{90}&\frac{-1}{480}\\\frac{117}{640}&\frac{27}{1280}&\frac{3}{128}&\frac{-9}{2560}\\\frac{4}{15}&\frac{1}{15}&\frac{4}{15}&0\end{bmatrix}$$

when $i=1$

$$e_1=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix},\ b_1=\begin{bmatrix}0&0&0&\frac{251}{2880}\\0&0&0&\frac{29}{360}\\0&0&0&\frac{27}{320}\\0&0&0&\frac{7}{90}\end{bmatrix},\ c_1=\begin{bmatrix}\frac{323}{1440}&\frac{-11}{120}&\frac{53}{1440}&\frac{-19}{2880}\\\frac{31}{90}&\frac{1}{15}&\frac{1}{90}&\frac{-1}{360}\\\frac{51}{160}&\frac{9}{40}&\frac{21}{160}&\frac{-3}{320}\\\frac{16}{45}&\frac{2}{15}&\frac{16}{45}&\frac{7}{90}\end{bmatrix}$$

Analysis of Basic Properties of the Method

Order of the Block

According to fatunla (1991) and lambert (1973) the truncation error associated with (2) is defined by

$$L\left[y(x);h\right]=\varphi_0(t)\tau_n+\varphi_{\frac{1}{2}}(t)\tau_{n+\frac{1}{2}}+h^2\left[\sum_{j=0}^{1}\psi_j(t)g_{n+j}+\sum_{i=\frac{1}{5}}^{\frac{4}{5}}\psi_i(t)g_{n+i}\right] \quad (10)$$

Assumed that $y(x)$ can be differentiated. Expanding (9) in Taylor's series and comparing the coefficient of h gives the expression

$$L\{y(x):h\}=C_0y(x)+C_1y'(x)+\ ,\ .\ .\ +C_ph^py^p(x)+C_{p+1}h^{p+1}y^{p+1}(x)$$

Where the constant coefficients are given below

$$C_0=\sum_{j=0}^{k}\varphi_j \quad , \quad C_1=\sum_{j=1}^{k} j\varphi_j$$

$$C_q=\frac{1}{q!}\sum_{j=0}^{k} j^q\varphi_j - q(q-1)\left(\sum_{j=0} j^{q-2}\psi_j+\left(\frac{1}{4}\right)^{q-2}\psi_{\frac{1}{4}}+\left(\frac{1}{2}\right)^{q-2}\psi_{\frac{1}{2}}+\left(\frac{3}{4}\right)^{q-2}\psi_{\frac{3}{4}}+1^{q-2}\psi_1\right), \quad q=2,3,4...$$

Definition 1: the linear operator and the associated continuous linear multistep method (10) are said to be of order p if $c_0=c_1=c_2=...=c_p=0$ $c_{p+1}=0$, *and* $c_{p+2}\neq 0$, c_{p+2} is called the error constant and the local truncation error is given by

For our method

Comparing the coefficient of h gives $C_0=C_1=C_2=C_3=...=C_6=0$ and

$$C_7=\left[\frac{-107}{13762560} \quad \frac{-1}{53760} \quad \frac{-27}{917504} \quad \frac{-1}{26880}\right]^T$$

Hence our method is of order five (5).

Consistency

The One-step Hybrid trigonometrically fitted second derivative is consistent since it has order is greater than or equal to one.

Zero Stability of Our Method

The One-Step One Hybrid Block trigonometrically fitted fourth derivative hybrid method is said to be zero-stable if as $h\to 0$, the root $z_i, i=0\left(\frac{1}{4}\right)1$ of the first characteristic polynomial $\rho(z)=0$ that is $\rho(z)=\det\left[\sum_{j=0}^{k} A^{(i)}z^{k-i}\right]=0$ Satisfies $|z_i|\leq 1$ and for those roots with $|z_i|=1$, multiplicity must not exceed two.

Convergency

The compulsory terminolgy for the trigonometrically fitted to be convergent is that they must be consistent and zero-stable. Hence, our method converges since all conditions are satisfied.

Linear Stability

According to Hairer and Wanner, the concept of A-satbility is discussed by applying the test equation

$$y^{(k)} = \lambda^{(k)} y$$

to yield

$$Y_m = \mu(z) Y_{m-1}, z = \lambda h$$

where is the amplification matrix given by

$$\mu(z) = (\zeta^0 - z\eta^{(0)} - z^2\eta^{(0)})^{-1}(\zeta^1 - z\eta^{(1)} - z^2\eta^{(1)})$$

The matrix $\mu(z)$ has eigen values $(0, 0, \cdots, \zeta_k)$ where ζ_k is called the stability function. Thus, ths stability function of our methodwith four off-grid points is given by

$$\xi = \frac{-16203z^4 + 255236z^3 + 212896z^2 - 12754944z + 26542080}{-20736z^4 + 241920z^3 + 262656z^2 - 12257280z + 26542080}$$

Region of Absolute Stability

The stability polynomial of our method is found to be

$$-h^4\left(\frac{1}{1280}w^4 + \frac{5401}{8847360}w^3\right) + h^3\left(\frac{7}{768}w^4 - \frac{91171}{6635520}\right) + h^2\left(\frac{19}{1920}w^4 + \frac{66143}{829440}w^3\right) - h\left(\frac{133}{288}w^4 - \frac{2071}{2880}w^3\right) + w^4 - 2w^3$$

3.3 Mathematical Computation of the method

Problem I We consider the stiff equation (Source: Adeniran *and Olanegan* (2019))

$y'' = -100y + 99\sin(x)$, $y(0) = 1$, $y'(0) = 11$ $h = \frac{1}{320}$

Exact Solution: $y(x) = \cos(10x) + \sin(10x) + \sin(x)$,

Table 1 Comparison of the proposed method with Adeniran *and Olanegan* (2019)

x-values	Error in our method	Error in [3]
0.1	8.91000E-17	9.99E-14
0.2	1.7580E-16	1.20 E-13
0.3	2.5970E-16	7.90E-13
0.4	3.4110E-16	1.69E-13
0.5	4.1920E-16	5.00E-13
0.6	4.9410E-16	2.00E-13
0.7	5.6600E-16	8.99 E-14
0.8	6.3460E-16	2.00 E-13
0.0	6.9940E-16	3.00E-13
1.0	7.6090E-16	2.99E-13

Problem II Consider the highly Oscilatory equation (source: Adeniran and Edaogbogun (2021))

$y'' = -\lambda 2 y$, $y(0) = 1$, $y'(0) = 2$, $\lambda = 2$, $h = 0.01$

Exact Solution: $y(x) = \cos 2x + \sin 2x$

x-values	Error in our method	Error in [11]
0.1	4.0000E-18	4.3881E-11
0.2	7.8000E-18	7.9019E-11
0.3	1.1500E-17	2.5525E-10
0.4	1.5200E-17	1.1525E-10
0.5	1.8800E-17	1.9079E-10
0.6	2.2400E-17	2.3002E-10
0.7	2.5900E-17	2.7014E-10
0.8	2.9300E-17	3.1112E-10
0.9	3.2700E-17	3.5291E-10
1.0	3.5800E-17	3.9545E-10

Conclusions

It is evident from the above tables that our proposed method has significant improvement over the existing methods. Trigonometric-Fitted One-Step 3 Points Hybrid Block Method is proposed for direct solution of general second order stiff and oscillatory problems where by it is self-starting when implemented. The developed method converges and is of order five.

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