African Multidisciplinary
Journal of Sciences and Artificial Intelligence

ISSN : 0000-0000
Index: Harvard, Boston, Sydney University, Dimensions, Lens, ResearchGet Scilit, Semantic, Google Scholar, Base etc

https://doi.org/10.58578/AMJSAI.v1i1.3369

Using the Residue Theorem to Compute Real-Valued Integrals

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Article Info:

Submitted:	Revised:	Accepted:	Published:
Jun 17, 2024	Jul 12, 2024	Jul 14, 2024	Jul 17, 2024

Abstract

Complex analysis is a branch of mathematics that studies complex numbers. Cauchy's residue theorem is a very important theorem in complex integral calculus. Many researchers have studied and tried applying it in different academic fields. In this paper, we studied the residue theorem and used it to compute some complicated real-valued integrals that appeared to be difficult in computing in the domain of real numbers. This is done by first converting the real-valued function to a complex-valued function, obtain the residue of the function and then apply the residue theorem to obtain the value of the integral.

Keywords: Residue, Function, Singularity, Integral

INTRODUCTION

The development of complex analysis seems to have been the direct result of mathematician's urge to generalize. It was systematically sought by analogy with real analysis. Mathematicians during the period of emergence of complex analysis tended to assumed that everything in real analysis must be meaningful in the complex case. In the study of complex analysis, complex integration is a very important and significant concept which serves as a powerful tool for not only mathematicians (Azram et al, 2013). Residue theorem (Cauchy's Residue Theorem) comprises of the Cauchy's integral theorem, Cauchy's integral formula and derivatives computed of high order formula (Lang, S. 1999).

When integrals are computed along a path that winds round points where the function becomes infinite, Cauchy computed those integrals using the theory of residues, which the latter requires just a computation of a constant known as the residue of the function at each exceptional point. The residue theorem serves as a basic theorem in complex integral calculus and is applied to solve problems in both real and complex calculus. It is also applied to other fields different from mathematics (Boshan Niu, 2023).

The theorem is used in solving problems on wind energy, equation of mechanical speed, electrical networks with exponential source and also in solving certain differential equations (Mizera, S. 2020 and Vidras, A., Yger, A. 2001).

DEFINITIONS

1. **Singularity:** A point z_0 at which a function f(z) fails to be analytic is called a singular point or a singularity of f(z).
2. **Analytic Function:** A function f(z) is said to be analytic at a point z_0 provided that its derivative exists at the point z_0 and its neighborhood.
3. **Laurent Series:** Laurent series about a point is a series representation of a function that fails to be analytic at that point. i.e. Suppose that a function f(z) is analytic throughout an annular domain $Q_1 < |z - z_0| < Q_2$ with center z_0, let C denote any positively oriented simple closed contour around z_0 and lying in that domain. Then at each point in the domain, f(z) has the series representation

   $$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=1}^{\infty} a_n (z - z_0)^n, \quad (Q_1 < |z - z_0| < Q_2) \qquad \ldots(1)$$

   Where the coefficients a_n and b_n are

   $$a_n = \frac{1}{2\pi i} \int_c \frac{f(z)dz}{(z - z_0)^{n+1}} \quad (n = 0, 1, 2, \ldots) \qquad \ldots(2)$$

$$b_n = \frac{1}{2\pi i}\int_c \frac{f(z)dz}{(z-z_0)^{-n+1}} \ (n = 1, 2, 3, \dots)$$

…(3)

4. **Residues:** consider the function f(z) with an isolated singularity at z_0, i.e. defined on $0 < |z - z_0| < r$ and with Laurent series

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=1}^{\infty} a_n (z - z_0)^n$$

…(4)

The residue of f at z_0 is b_1. This is denoted as $Res(f,\ z_0)$ or $\operatorname*{Res}_{z\to z_0} f = b_1$.

Residue Theorem (Cauchy's Residue Theorem)

Theorem 1

Let C be a simple closed contour, positively oriented. If a function g(z) is analytic inside and on C except at some finite number of singular points z_p $(p = 0, 1, 2, \dots, n)$ inside C (as in the figure below), then

$$\int_c g(z)dz = 2\pi i \sum_{p=1}^{n} \operatorname{Re} s(g, z_p)$$

…(5)

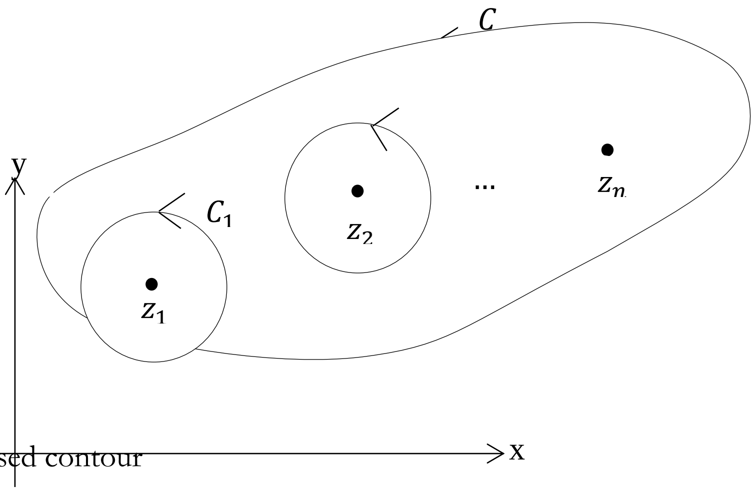


Figure 1: A simple closed contour

Proof

Supposing the points z_p $(p = 0, 1, 2, \dots, n)$ are centers of the positively oriented circles c_p which completely lies inside C having no points in common. The function g(z) is analytic inside and on the boundary of the circles c_p and c. Then by Cauchy-Goursat theorem we have that

$$\int_c g(z)dz = 0 \qquad \ldots(6)$$

But then

$$\int_c g(z)dz = \int_{c_1} g(z)dz + \int_{c_2} g(z)dz + \cdots + \int_{c_n} g(z)dz \qquad \ldots(7)$$

$$\int_c g(z)dz = \sum_{p=1}^{n} \int_{c_p} g(z)dz \qquad \ldots(8)$$

$$\int_c g(z)dz - \sum_{p=1}^{n} \int_{c_p} g(z)dz = 0 \qquad \ldots(9)$$

By hypothesis

$$2\pi i \sum_{p=1}^{n} \operatorname{Re} s(g, z_p) - \sum_{p=1}^{n} \int_{c_p} g(z)dz = 0 \qquad \ldots(10)$$

$$2\pi i \sum_{p=1}^{n} \operatorname{Re} s(g, z_p) = \sum_{p=1}^{n} \int_{c_p} g(z)dz$$

$$2\pi i \sum_{p=1}^{n} \operatorname{Re} s(g, z_p) = \int_{c_p} g(z)dz \qquad \ldots(11)$$

$$\int_{c_p} g(z)dz = 2\pi i \operatorname{Re} s(g, z_p) \text{ where } p = 1, 2, 3, \ldots, n \qquad \ldots(12)$$

This completes the proof.

Theorem 2

Suppose g(z) is defined in the upper half-plane. If there is an $\alpha > 1$ and $Q > 0$ such that $|g(z)| < \frac{Q}{|z|^{\alpha}}$ for $|z|$ large then $\lim_{T\to\infty}\int_{c_T} g(z)dz = 0$, where c_T is a semicircle $Te^{i\theta}$, $0 < \theta < \pi$ as shown below;

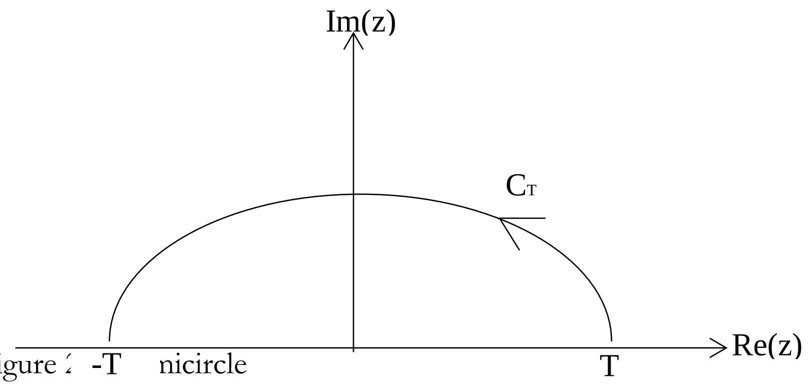


Figure 2 nicircle

Theorem 3

For a real-valued function g(x), the definite integral of g(x) from the negative infinity to positive infinity is

$$\int_{-\infty}^{\infty} g(x)dx = 2\pi i \sum residues\ of\ g(x) in\ the\ upper\ half-plane \quad \ldots(13)$$

Theorem 4

If g(z) has a simple zero at z_0 then $\frac{1}{g(z)}$ has a simple pole at z_0 and $Res\left(\frac{1}{g(z)}, z_0\right) = \frac{1}{g'(z_0)}$

APPLICATION

Example 1

Compute the integral $\int_{-\infty}^{\infty} g(x)dx$ such that $g(x) = \frac{1}{x^6+1}$.

Solution

It is difficult to find the integral of $\frac{1}{x^6+1}$. We use the residue theorem to compute the result. Set a complex function g(z) such that $g(z)=\frac{1}{z^6+1}$

We find the poles of g(z). Note for g(z) not to exists, the denominator must be zero i.e.

$z^6+1=0$.

Now, we obtain the values of z as

$z^6+1=0 \Rightarrow z=\sqrt[6]{-1}=\sqrt[6]{e^{\pi i}}=e^{\frac{\pi i}{6}}\cdot e^{\frac{2mm\pi i}{6}}$ (where m = 1, 2, 3, 4, 5)

$$z_0 = e^{\frac{\pi i}{6}}\cdot e^{\frac{2(0)\pi i}{6}} = e^{\frac{\pi i}{6}}$$

$$z_1 = e^{\frac{\pi i}{6}}\cdot e^{\frac{2(1)\pi i}{6}} = e^{\frac{\pi i}{2}}$$

$$z_2 = e^{\frac{\pi i}{6}}\cdot e^{\frac{2(2)\pi i}{6}} = e^{\frac{5\pi i}{6}}$$

…(14)

$$z_3 = e^{\frac{\pi i}{6}}\cdot e^{\frac{2(3)\pi i}{6}} = e^{\frac{7\pi i}{6}}$$

$$z_4 = e^{\frac{\pi i}{6}}\cdot e^{\frac{2(4)\pi i}{6}} = e^{\frac{9\pi i}{6}}$$

$$z_5 = e^{\frac{\pi i}{6}}\cdot e^{\frac{2(5)\pi i}{6}} = e^{\frac{11\pi i}{6}}$$

Observe that points z_0, z_1 and z_2 lies in the upper half plane, we apply equation (13) and therefore find the residue of the function f(z) over these points. Let the complex function

$w(x)=z^6+1$

Substituting z_0, z_1 and z_2, we get

$$w(z_0) = z_0^6 + 1 = \left(e^{\frac{\pi i}{6}}\right)^6 + 1 = e^{\pi i} + 1 = -1 + 1 = 0$$

$$w(z_1) = z_1^6 + 1 = \left(e^{\frac{\pi i}{2}}\right)^6 + 1 = e^{3\pi i} + 1 = -1 + 1 = 0$$

…(15)

$$w(z_2) = z_2^6 + 1 = \left(e^{\frac{5\pi i}{6}}\right)^6 + 1 = e^{5\pi i} + 1 = -1 + 1 = 0$$

But then taking their derivative we have

$$w'(z_0) = 6\left(e^{\frac{\pi i}{6}}\right)^5, \; w'(z_1) = 6\left(e^{\frac{\pi i}{2}}\right)^5, \quad w'(z_2) = 6\left(e^{\frac{5\pi i}{6}}\right)^5$$

...(16)

Calculating the residue of f(z) over z_0, z_1 and z_2 using theorem 3.4

$$Res_{z_0} f(z) = Res_{z_0} \frac{1}{w(z_0)} = \frac{1}{w'(z_0)} = \frac{1}{6\left(e^{\frac{\pi i}{6}}\right)^5} = \frac{1}{6} e^{-\frac{5\pi i}{6}}$$

$$Res_{z_1} f(z) = Res_{z_1} \frac{1}{w(z_1)} = \frac{1}{w'(z_1)} = \frac{1}{6\left(e^{\frac{\pi i}{2}}\right)^5} = \frac{1}{6} e^{-\frac{5\pi i}{2}}$$

...(17)

$$Res_{z_2} f(z) = Res_{z_2} \frac{1}{w(z_2)} = \frac{1}{w'(z_2)} = \frac{1}{6\left(e^{\frac{5\pi i}{6}}\right)^5} = \frac{1}{6} e^{-\frac{25\pi i}{6}}$$

We now evaluate the definite integral of real-valued function f(x) using the equation

$$\int_{-\infty}^{\infty} g(x)dx = 2\pi i \; \sum residues\ of\ g(x) in\ the\ upper\ half - plane$$

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \left[\frac{1}{6} e^{-\frac{5\pi i}{6}} + \frac{1}{6} e^{-\frac{5\pi i}{2}} + \frac{1}{6} e^{-\frac{25\pi i}{6}}\right]$$

$$= \frac{2\pi i}{6}\left[\left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right) + \left(\cos\frac{5\pi}{2} - i\sin\frac{5\pi}{2}\right) + \left(\cos\frac{25\pi}{6} - i\sin\frac{25\pi}{6}\right)\right]$$

$$= \frac{\pi i}{3}\left[\left(\frac{-\sqrt{3}}{2} - \frac{i}{2}\right) + (0 - i) + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)\right]$$

$$= \frac{\pi i}{3}[-2i]$$

$$= \frac{2\pi}{3}$$

$$\int_{-\infty}^{\infty} f(x)dx = \frac{2\pi}{3}$$

$$\int_{-\infty}^{\infty} \frac{1}{x^6 + 1} dx = \frac{2\pi}{3}$$

Example 2:

Show that $\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + b^2} dx = \frac{\pi e^{-b}}{2b}$

Where b > 0

Solution

Let $g(x) = \frac{cosx}{x^2+b^2} \Rightarrow \int_0^{\infty} g(x)dx$

The hypothesis of theorem 2 is not satisfied since $cosx$ goes to infinity in the half-plane. Replace $cosx$ with e^{ix} i.e.

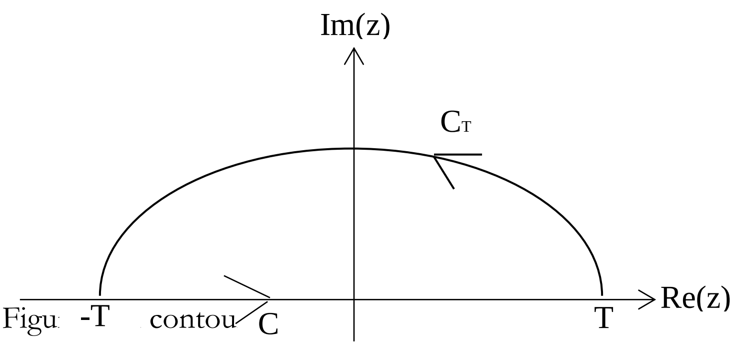
$I^* = \int_{-\infty}^{\infty} \frac{cosx}{x^2+b^2} dx$ with $I = \frac{1}{2} Re(I^*)$
…(18)

Now, let

Let $g(x) = \frac{e^{iz}}{z^2+b^2}$, for Let $z = x + iy$ and y > 0.

Let $|g(z)| = \left|\frac{e^{iz}}{z^2+b^2}\right| = \frac{|e^{iz}|}{|z^2+b^2|} = \frac{|e^{i(x+iy)}|}{|z^2+b^2|} = \frac{e^{-y}}{|z^2+b^2|}$
…(19)

Since $e^{-y} < 1$, implies that g(z) satisfies the hypothesis of theorem 2 in the upper half plane. Now considering the contour



Figu contou

We have

$$\lim_{T \to \infty} \int_{C_T} g(z)dz = 0 \quad \text{and}$$

$$\lim_{T\to\infty}\int_C g(z)dz = \int_{-\infty}^{\infty} g(z)dz = I^*$$

$\ldots(20)$

By residue theorem

$$I^* = \lim_{T\to\infty}\int_{C+C_T} g(z)dz = 2\pi i\sum residues\ of\ g(z)\ inside\ the\ contour$$

g(z) has simple poles at $\pm bi$ and lies inside the contour. We compute the residue as a limit using L'Hospital's rule

$$\operatorname{Re}s\big(g(z),bi\big)=\lim_{z\to bi}(z-bi)\frac{e^{iz}}{z^2+b^2}=\lim_{z\to bi}\frac{i(z-bi)e^{iz}+e^{iz}}{2z}=\frac{0+e^{-y}}{2bi}=\frac{e^{-y}}{2bi}$$

$$I^* = \operatorname{Re}s\big(g(z),bi\big)=2\pi i\times\frac{e^{-b}}{2bi}=\frac{\pi e^{-b}}{b}$$

Implying

$$I=\frac{1}{2}\operatorname{Re}(I^*)=\frac{1}{2}\frac{\pi e^{-b}}{b}=\frac{\pi e^{-b}}{2b}\ \text{ as claimed}$$

CONCLUSION

The residue theorem serves as a very useful tool for not only computing the integrals of complex-valued function but also in solving real-valued integrals, which appears to be difficult to compute in the real number calculus. The residue theorem is used to compute some real-valued integrals as seen in example one and two which makes the computation much easier.

The residue theorem can be used in other fields different from mathematics in future and base on the fact that life is based on real numbers, applying the residue theorem is challenging and needs further study.

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