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RATIONAL CONTRACTION IN METRIC SPACE AND COMMON FIXED POINT THEOREMS

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Abstract

The study of contraction mappings in fixed-point theory is a fascinating and crucial field of mathematics. The concept of contraction plays a vital role in proving the existence and uniqueness of fixed points. Banach's contraction theory offers a fixed point theorem that is widely accepted as unique in most analyses. By using rational expressions in metric spaces, we can achieve unique results in general contraction mapping. These results are based on several innovative ideas stemming from the latest research. The delivered results upgrade and federate many existing outcomes on the topic in the literature Bhardwaj, R. et al. (2007) Chouhan et al. (2014) and Garg and Priyanka (2016). Also gives some suitable examples for verifying our results.

Keywords: Complete Metric space, Fixed point, Common Fixed Point, Rational expressions

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INTRODUCTION

The literature of fixed point (FP) for contraction mapping embarked upon Banach contraction principle (BCP) in 1922 and known as thse Banach fixed point Theorem (BFP. BCP is stated as follows:

"A single valued contractive type mapping on a complete metric space has a unique FP."

The fixed point theorem can be used to prove that a differential equation, an integral equation, or a partial differential equation has a unique solution. They are utilized in the investigation of problems related to the optimal control of these systems.

Outcomes of FP (fixed point) treatment with general contractive conditions relating to rational type expression are also interesting. Some notorious results from this track are included (see [Das and Gupta (1975), Khan, M.S.(1976), Jaggi, D.S.(1977), Fisher, B. (1978), Jaggi and Das (1980), Bhardwaj et al. (2007), Dutta and Chaudhary (2008), Jieli, et al.(2012), Shrivastava, et al.(2013), Karapinar,E., Shatanawi, W. and Tas, K (2013), Saluja et al.(2015), Garg, M.(2015), Garg, M. and Priyanka(2016), Karapinar,E., et al. (2016), Fulga, A.(2018), Gupta, S. and Garg, A.K., (2019), Fulga, A. (2020)]). It has been extended and generalized various types of contraction in different metric spaces.

Dewangan,S. and Tiwari, S.K.(2020), extended, generalized and improved the result of Shriastava et al (2014) and established a unique fixed point theorem for expansive mapping in metric space satisfying rational expression . In 2021, Rao S. and Kalyani, obtained fixed point results partially ordered metric space with rational expression. Subsequently, Rao,. N.S. et al. (2021), demonstrated result of fixed point of monotone functions in ordered metric space.

Furthermore, Raji, M. and Adegboye, F. (2022), proved the existence and uniqueness of some fixed point for nonlinear contractive mappings with rational expressions in the context of metric space endowed with partial order. Most recently, Chaodary et al. (2023) established a result in the fixed point theory for multivalued mizoguchi-Takahashi type rational contraction in metric space.

In this manuscript, we extend, generalize, and improve earlier work by Bhardwaj, R. et al. (2007), Chouhan et al. (2014), and Garg and Priyanka (2016) to determine in common fixed-point theorems for rational contraction on metric space.

PRELIMINARIES NOTES

We recall the definition of metric space and their properties as follows:

Definition 1 [Shrivastava et al. (2014.)]: Infer that $\Im \neq \phi$ and $d: \Im \times \Im \to [0, \infty)$ be a mapping that known as metric on $\Im$, if satisfies the following conditions:

(i) $d(u, v) \geq 0 \; \forall u, v \in \Im$;

(ii) $d(u, v) = 0 \Leftrightarrow u = v \;\; \forall u, v \in \Im$;

(iii) $d(u, v) = d(v, u) \; u, v \in \Im$;

(iv) $d(u, v) \leq d(u, w) + d(w, v)$, for all $u, v, w \in \Im$.

Then, we can refer to the pair $(\Im, d)$ as a metric space. .

Definition 2[Shrivastava et al. (2014.)]: Assume that $\{u_n\}$ is a sequence referred to Cauchy **s**equence in $(\Im, d)$, even if $\varepsilon > 0 \; \exists \; n_0 \in N$, like that, $\forall \;\; m, n > n_0$

$$d(u_n, u_m) < \varepsilon. i. e. \lim_{n \to \infty} d(u_n, u_m) < \varepsilon.$$

Definition 3 [Shrivastava et al. (2014.)]: The sequence $\{u_n\}$ is convergent to $u \in \Im$ in $(\Im, d)$, if $\lim_{n \to \infty} d(u_n, u) = 0, \quad$ or $\; u_n \to u$.

Definition 4[Shrivastava et al. (2014)]: when everything the Cauchy sequences converges in $(\Im, d)$ then metric space$(\Im, d)$ is so called complete.

MAIN RESULTS

Theorem 5. Consider that, $(\Im, \leq)$ be a ordered set of partially and let ∂ is a metric on $(\Im, \partial)$,where $(\Im, \partial)$ is a complete metric space. Take over, $j, k: \Im \to \Im$ be a continuous self mapping such that $v, \xi \in \Im$, $v \neq \xi$ there exist $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5 \in [0,1)$ with $\theta_1 + \theta_2 + \theta_3 + \theta_4 + 2\theta_5 < 2$, satisfies the following condition

$$\partial(jv, k\xi) \leq \theta_1 \frac{\partial(v, jv).\partial(\xi, k\xi) + \partial(v, k\xi).\partial(\xi, jv)}{\partial(v, jv) + \partial(\xi, k\xi) + \partial(v, k\xi) + \partial(\xi, jv)} + \theta_2 \frac{\partial(v, jv).\partial(\xi, jv) + \partial(\xi, k\xi).\partial(v, k\xi)}{\partial(v, jv) + \partial(\xi, jv) + \partial(\xi, k\xi) + \partial(v, k\xi)}$$

$$+ \theta_3 \frac{\partial(v, jv).\partial(v, jv) + \partial(v, k\xi).\partial(\xi, jv)}{\partial(v, jv) + \partial(v, jv) + \partial(v, k\xi) + \partial(\xi, jv)} + \theta_4 \frac{\partial(v, jv).\partial(v, k\xi) + \partial(\xi, jv).\partial(\xi, k\xi)}{\partial(v, jv) + \partial(v, k\xi) + \partial(\xi, jv) + \partial(\xi, k\xi)}$$

$$+ \theta_5 \partial(v, \xi). \qquad (5.1)$$

For $\nu_0 \in \mathcal{H}$, suppose $\{\nu_k\}_{k=0}^{\infty} \subset \mathfrak{J}$ and defined by $\nu_{2k+1} = \jmath\nu_{2k}$ and $\nu_{2k+2} = \hbar\nu_{2k+1}$ $k = 0,1,2$ .... be the Picard iteration association to $\jmath$ and $\hbar$. Then there exist $\nu_k \to \nu^*, \lim_{k\to\infty} \jmath^k\nu = \nu^* = \lim_{k\to\infty} \xi$. Then ν^* is a unique common fixed point.

Proof: Let $\nu_0 \in \mathcal{H}$,we define the iterative sequence$\{\nu_{2k}\}$ in $\mathfrak{J}$ defined as

$$\nu_{2k+1} = \jmath\nu_k = \jmath^{2k+1}\nu_0 ,$$

And

$$\nu_{2k+2} = \hbar\nu_{2k+1} = \hbar^{2k+2}\nu_0, \text{ for every } k \geq 1.$$

Now put $\nu = \nu_{2k}$ and $\xi = \nu_{2k-1}$ in (5.1) we have

$$\begin{aligned}
\partial(\nu_{2k+1}, \nu_{2k}) &= \partial(\jmath\nu_{2k}, \hbar\nu_{2k-1}) \\
&\leq \theta_1 \frac{\partial(\nu_{2k},\ \jmath\nu_{2k}).\partial(\nu_{2k-1},\ \hbar\nu_{2k-1}) + \partial(\nu_{2k},\ \hbar\nu_{2k-1}).\partial(\nu_{2k-1},\ \jmath\nu_k)}{\partial(\nu_{2k},\ \jmath\nu_{2k}) + \partial(\nu_{2k-1},\ \hbar\nu_{2k-1}) + \partial(\nu_{2k},\ \hbar\nu_{2k-1}) + \partial(\nu_{2k-1},\ \jmath\nu_k)} \\
&+ \theta_2 \frac{\partial(\nu_{2k},\ \jmath\nu_{2k}).\partial(\nu_{2k-1},\ \jmath\nu_{2k}) + \partial(\nu_{2k-1},\ \hbar\nu_{2k-1}).\partial(\nu_{2k},\ \hbar\nu_{2k-1})}{\partial(\nu_{2k},\ \jmath\nu_{2k}) + \partial(\nu_{2k-1},\ \jmath\nu_{2k}) + \partial(\nu_{k-1},\ \hbar\nu_{2k-1}) + \partial(\nu_k,\ \hbar\nu_{2k-1})} \\
&+ \theta_3 \frac{\partial(\nu_{2k},\ \jmath\nu_{2k}).\partial(\nu_{2k},\ \jmath\nu_k) + \partial(\nu_{2k},\ \hbar\nu_{2k-1}).\partial(\nu_{2k-1},\ \jmath\nu_{2k})}{\partial(\nu_{2k},\ \jmath\nu_{2k}) + \partial(\nu_{2k},\ \jmath\nu_{2k}) + \partial(\nu_k,\ \hbar\nu_{2k-1}) + \partial(\nu_{k-1},\ \jmath\nu_k)} \\
&+ \theta_4 \frac{\partial(\nu_{2k},\ \jmath\nu_{2k}).\partial(\nu_k,\ \hbar\nu_{2k-1}) + \partial(\nu_{2k-1},\ \jmath\nu_{2k}).\partial(\nu_{2k-1},\ \hbar\nu_{2k-1})}{\partial(\nu_{2k},\ \jmath\nu_{2k}) + \partial(\nu_{2k},\ \hbar\nu_{2k-1}) + \partial(\nu_{2k-1},\ \jmath\nu_{2k}) + \partial(\nu_{k-1},\ \hbar\nu_{2k-1})} \\
&+ \theta_5\partial(\nu_{2k}, \nu_{2k-1}). \\
&= \theta_1 \frac{\partial(\nu_{2k},\nu_{2k+1}).\partial(\nu_{2k-1},\nu_{2k}) + \partial(\nu_{2k},\nu_{2k}).\partial(\nu_{2k-1},\nu_{2k+1})}{\partial(\nu_{2k},\nu_{2k+1}) + \partial(\nu_{2k-1},\nu_{2k}) + \partial(\nu_{2k},\nu_{2k}) + \partial(\nu_{2k-1},\nu_{2k+1})} \\
&+ \theta_2 \frac{\partial(\nu 2_k,\nu_{2k+1}).\partial(\nu_{2k-1},\nu_{2k+1}) + \partial(\nu_{2k-1},\nu_{2k}).\partial(\nu_{2k},\nu_{2k})}{\partial(\nu_{2k},\nu_{2k+1}) + \partial(\nu_{2k-1},\nu_{2k+1}) + \partial(\nu_{2k-1},\nu_{2k}) + \partial(\nu_{2k},\nu_{2k})} \\
&+ \theta_3 \frac{\partial(\nu_{2k},\nu_{2k+1}).\partial(\nu_{2k},\nu_{2k+1}) + \partial(\nu_{2k},\nu_{2k}).\partial(\nu_{2k-1},\nu_{2k+1})}{\partial(\nu_{2k},\nu_{2k+1}) + \partial(\nu_{2k},\nu_{2k+1}) + \partial(\nu_{2k},\nu_{2k}) + \partial(\nu_{2k-1},\nu_{2k+1})} \\
&+ \theta_4 \frac{\partial(\nu_{2k},\nu_{2k+1}).\partial(\nu_{2k},\nu_{2k}) + \partial(\nu_{2k-1},\nu_{2k+1}).\partial(\nu_{2k-1},\nu_{2k})}{\partial(\nu_{2k},\nu_{2k+1}) + \partial(\nu_{2k},\nu_{2k}) + \partial(\nu_{2k-1},\nu_{2k+1}) + \partial(\nu_{2k-1},\nu_{2k})} \\
&+ \theta_5\partial(\nu, \nu_{k-1}). \\
&\leq \left(\frac{\theta_1}{2} + \frac{\theta_2}{2} + \frac{\theta_3}{2}\right)\partial(\nu_{2k}, \nu_{2k+1}) + \left(\frac{\theta_4}{2} + \theta_5\right)\partial(\nu_{2k-1}, \nu_{2k}).
\end{aligned}$$

Therefore,

$$\partial(\nu_{2k+1}, \nu_{2k}) \leq \pi\partial(\nu_{2k-1}, \nu_{2k}).$$

Where $\pi = \frac{\left(\frac{\theta_4}{2}+\theta_5\right)}{\left(1-\frac{\theta_1}{2}-\frac{\theta_2}{2}-\frac{\theta_3}{2}\right)} < 1$, but $\theta_1 + \theta_2 + \theta_3 + \theta_4 + 2\theta_5 < 2$.

So, continuing this process, we get

$$\partial(\nu_{2k+1}, \nu_{2k}) \leq \pi^{2k}\partial(\nu_0, \nu_1).$$

Now for $l > k$ and using the triangle inequality, we obtain

$$\begin{aligned}\partial(\nu_{2k}, \nu_{2l}) &\leq \partial(\nu_{2k}, \nu_{2k+1}) + \partial(\nu_{2k+1}, \nu_{2k+2}) + \cdots + \partial(\nu_{2l-1}, \nu_{2l}) \\ &\leq \pi^{2k}\partial(\nu_0, \nu_1) + \pi^{2k+1}\partial(\nu_0, \nu_1) + \cdots + \pi^{2k+2l-1}\partial(\nu_0, \nu_1) \\ &\leq (\pi^{2k} + \pi^{2k+1} + \cdots + \pi^{2k+2l-1})\partial(\nu_0, \nu_1) \\ &\leq \frac{\pi^{2k}}{1-\pi}\,\partial(\nu_0, \nu_1) \to \partial(\nu_k, \nu_l) \to 0,\ k, l \to \infty.\end{aligned}$$

Hence in $\mathfrak{J}$, $\{\nu_{2k}\}$ is a Cauchy sequence. Due to the fact that $(\mathfrak{J}, \partial)$ is a complete metric spaces, $\exists$ a point $\nu^* \in \mathfrak{J}$ Such that $\nu_{2k} \to \nu^*$ as $k \to \infty$. Again, since $\jmath$ is a continuous such that $\jmath(\nu) = \jmath(\lim_{k\to\infty} \nu_{2k}) = \lim_{k\to\infty} \jmath\nu_{2k} = \lim_{k\to\infty} \nu_{2k+1} = \nu^*$.

Hence ν^* is fixed point of $\jmath$ in $\mathfrak{J}$. Similarly, we can show that $\hbar(\xi) = \nu^*$. Since $\jmath(\nu) = \nu^* = \hbar(\xi)$. Therefore, ν^* is a common fixed point of $\jmath$ *and* $\hbar$.

Now we will show that ν^* is a unique common fixed point of $\jmath$ and $\hbar$ in $\mathfrak{J}$. Suppose ξ^* is another common fixed point of $\jmath$ *and* $\hbar$ in $\mathfrak{J}$. then we have

$$\begin{aligned}\partial(\nu^*, \xi^*) &= \partial(\jmath\nu^*, \hbar\xi^*) \\ &\leq \theta_1 \frac{\partial(\nu^*, \jmath\nu^*).\partial(\xi^*, \hbar\xi^*) + \partial(\nu^*, \hbar\xi^*).\partial(\xi^*, \jmath\nu^*)}{\partial(\nu^*, \jmath\nu^*) + \partial(\xi^*, \hbar\xi^*) + \partial(\nu^*, \hbar\xi^*) + \partial(\xi^*, \jmath\nu^*)} \\ &+ \theta_2 \frac{\partial(\nu^*, \jmath\nu^*).\partial(\xi^*, \jmath\nu^*) + \partial(\xi^*, \hbar\xi^*).\partial(\nu^*, \hbar\xi^*)}{\partial(\nu^*, \jmath\Gamma\nu^*) + \partial(\xi^*, \jmath\nu^*) + \partial(\xi^*, \hbar\xi^*) + \partial(\nu^*, \hbar\Gamma\xi^*)} \\ &+ \theta_3 \frac{\partial(\nu^*, \jmath\nu^*).\partial(\nu^*, \jmath\nu^*) + \partial(\nu^*, \hbar\xi^*).\partial(\xi^*, \jmath\nu^*)}{\partial(\nu^*, \jmath\nu^*) + \partial(\nu^*, \jmath\nu^*) + \partial(\nu^*, \hbar\xi^*)\partial(\xi^*, \jmath\nu^*)} \\ &+ \theta_4 \frac{\partial(\nu^*, \jmath\nu^*).\partial(\nu^*, \hbar\xi^*) + \partial(\xi^*, \jmath\nu^*).\partial(\xi^*, \hbar\xi^*)}{\partial(\nu^*, \jmath\nu^*).\partial(\nu^*, \hbar\xi^*) + \partial(\xi^*, \jmath\nu^*).\partial(\xi^*, \hbar\xi^*)} \\ &+ \theta_5\partial(\nu^*, \xi^*). \\ &= \theta_1 \frac{\partial(\nu^*, \nu^*).\partial(\xi^*, \xi^*) + \partial(\nu^*, \xi^*).\partial(\xi^*, \nu^*)}{\partial(\nu^*, \nu^*) + \partial(\xi^*, \xi^*) + \partial(\nu^*, \xi^*) + \partial(\xi^*, \nu^*)} \\ &+ \theta_2 \frac{\partial(\nu^*, \nu^*).\partial(\xi^*, \nu^*) + \partial(\xi^*, \xi^*).\partial(\nu^*, \xi^*)}{\partial(\nu^*, \nu^*) + \partial(\xi^*, \nu^*) + \partial(\xi^*, \xi^*) + \partial(\nu^*, \xi^*)} \\ &+ \theta_3 \frac{\partial(\nu^*, \nu^*).\partial(\nu^*, \nu^*) + \partial(\nu^*, \xi^*).\partial(\xi^*, \nu^*)}{\partial(\nu^*, \nu^*) + \partial(\nu^*, \nu^*) + \partial(\nu^*, \xi^*)\partial(\xi^*, \nu^*)} \\ &+ \theta_4 \frac{\partial(\nu^*, \nu^*).\partial(\nu^*, \xi^*) + \partial(\xi^*, \nu^*).\partial(\xi^*, \xi^*)}{\partial(\nu^*, \nu^*).+\partial(\nu^*, \xi^*) + \partial(\xi^*, \nu^*) + \partial(\xi^*, \xi^*)} \\ &+ \theta_5\partial(\nu^*, \xi^*).\end{aligned}$$

Thus $\partial(\nu^*, \xi^*) \leq \left[\frac{\theta_1}{2} + \frac{\theta_3}{2} + \frac{\theta_4}{2} + \theta_5\right]\partial(\nu^*, \xi^*) \to 0$, which is a contradiction, because $\theta_1 + \theta_2 + \theta_3 + \theta_4 + 2\theta_5 < 2$. So, $\partial(\nu^*, \xi^*) = 0$. Implies that $\nu^* = \xi^*$, thus,ν^*

is a unique common fixed point of j and k in $\mathfrak{J}$. The theorem's proof approach has become done.

Example 6: Let $\mathfrak{J}$= [0, 1] with partial order " $\leq$ " and let $j, k: \mathfrak{J} \to \mathfrak{J}$ be defined by

$$j(v) = \begin{cases} \frac{1}{2} & \text{if} \in [0, \frac{1}{4}) \\ & = k(\xi) \\ v - \frac{1}{4} & \text{if } v \in [\frac{1}{4}, 1) \end{cases}$$

To show that j and k satisfies the contractive condition 51) of theorem (5), for $v = \frac{1}{2}$, $\xi = \frac{1}{6}, \theta_1 = \theta_2 = \theta_3 = \theta_4 = 1$. Then we have

$$jv = \frac{1}{4}, k\xi = \frac{1}{2}, \partial(v, jv) = \frac{1}{4}, \partial(\xi, k\xi) = \frac{1}{3}, \partial(v, k\xi) = 0, \partial(\xi, jv) = \frac{1}{12},$$

$$\partial(v, \xi) = \frac{1}{3} \text{ and } \partial(jv, k\xi) = \frac{1}{4}.$$

Thus, from (5.1), we get

$$\frac{1}{4} = \partial(jv, k\xi)$$

$$\leq \frac{2}{3} + \frac{1}{32} + \frac{3}{28} + \frac{1}{24} + \frac{1}{3}\theta_5.$$

$$= \frac{448+21+72+28}{672} + \frac{1}{3}\theta_5$$

$$\frac{1}{4} = \frac{569}{672} + \frac{1}{3}\theta_5 .$$

It follows that, $\theta_5 \geq \frac{401}{672}$. Hence all the conditions of Theorem 3.1 are satisfied and $\frac{1}{2}$ is a unique common fixed point of j and k.

Remark: If $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0$ and $\theta_5 = \alpha$, then the theorem reduce to Banach, S. (1922).

Theorem 7: Consider that, $(\mathfrak{J}, \leq)$ be a ordered set of partially and let ∂ is a metric on $(\mathfrak{J}, \partial)$,where $(\mathfrak{J}, \partial)$ is a complete metric space. Take over, $j, k: \mathfrak{J} \to \mathfrak{J}$ be a continuous self mapping such that $v, \xi \in \mathfrak{J}$, $v \neq \xi$ there exist $\theta_1, \theta_2, \theta_3, \theta_4, \in [0,1)$ with $2\theta_1 + 2\theta_2 + \theta_4 < 2$, satisfies the following condition

$$\partial(jv, k\xi) \leq \theta_1 \partial(v, \xi) + \theta_2 \frac{\partial(v, jv).\partial(\xi, k\xi) + \partial(v, k\xi).\partial(\xi, jv)}{\partial(v, \xi)}$$

$$+\theta_3 \frac{\partial(v, jv)[.\partial(v, jv) + \partial(\xi, k\xi)]}{\partial(v, \xi) + \partial(\xi, jv) + \partial(\xi, k\xi)}$$

$$+\theta_4 \frac{\partial(\nu,\ \jmath\nu).\partial(\xi,\ \jmath\nu)+\partial(\xi,\ \hbar\xi).\partial(\nu,\ \hbar\xi)}{\partial(\nu,\ \jmath\nu)+\partial(\xi,\ \jmath\nu)+\partial(\nu,\ \hbar\xi)}. \qquad (7.1)$$

For $\nu_0 \in \mathcal{H}$, suppose $\{\nu_k\}_{k=0}^{\infty} \subset \mathcal{H}$ and defined by $\nu_{2k+1} = \jmath\nu_{2k}$ and $\nu_{2k+2} = \hbar\nu_{2k+1}$ $k = 0,1,2$ .... be the Picard iteration association to $\jmath$ and $\hbar$. Then there exist $\nu_k \to \nu^*, \lim_{k\to\infty} \jmath^k\nu = \nu^* = \lim_{k\to\infty} \xi$. Then ν^* is a unique common fixed point.

Proof: : Let $\nu_0 \in \mathfrak{J}$,we define the iterative sequence$\{\nu_{2k}\}$ in $\mathfrak{J}$ defined as

$$\nu_{2k+1} = \jmath\nu_k = \jmath^{2k+1}\nu_0 ,$$

And

$$\nu_{2k+2} = \hbar\nu_{2k+1} = \hbar^{2k+2}\nu_0, \text{ for every } k \geq 1.$$

Now put $\nu = \nu_{2k}$ and $\xi = \nu_{2k-1}$ in (7.1) we have

$$\partial(\nu_{2k+1}, \nu_{2k}) = \partial(\jmath\nu_k, \hbar\nu_{2k-1})$$

$$\leq \theta_1\partial(\nu_{2k}, \nu_{2k-1}) + \theta_2 \frac{\partial(\nu_{2k},\ \jmath\nu_{2k}).\partial(\nu_{2k-1},\hbar\nu_{2k-1})+\partial(\nu_{2k},\ \hbar\nu_{2k-1}).\partial(\nu_{2k-1},\ \jmath\nu_{2k})}{\partial(\nu_{2k},\nu_{2k-1})} + \theta_3 \frac{\partial(\nu_{2k},\ \hbar\nu_{2k-1})[\partial(\nu_{2k},\ \jmath\nu_{2k})+\partial(\nu_{2k-1},\ \hbar\nu_{k-1})]}{\partial(\nu_{2k},\nu_{2k-1})+\partial(\nu_{2k-1},\ \hbar\nu_{2k-1})+\partial(\nu_{2k-1},\ \jmath\nu_k)} + \theta_4 \frac{\partial(\nu_{2k},\ \jmath\nu_k).\partial(\nu_{2k-1},\ \jmath\nu_{2k})+\partial(\nu_{k-1},\ \hbar\nu_{2k-1}).\partial(\nu_{2k},\ \hbar\nu_{2k-1})}{\partial(\nu_{2k},\ \jmath\nu_{2k})+\partial(\nu_{2k-1},\ \jmath\nu_k)+\partial(\nu_{2k-1},\ \hbar\nu_{2k-1})+\partial(\nu_{2k},\ \hbar\nu_{2k-1})}.$$

$$= \theta_1\partial(\nu_{2k}, \nu_{2k-1}) + \theta_2 \frac{\partial(\nu_{2k},\nu_{2k+1}).\partial(\nu_{2k-1},\nu_{2k})+\partial(\nu_{2k},\nu_{2k}).\partial(\nu_{2k-1},\nu_{2k+1})}{\partial(\nu_{2k},\nu_{2k-1})} + \theta_3 \frac{\partial(\nu_{2k},\nu_{2k})[\partial(\nu_{2k},\nu_{2k+1})+\partial(\nu_{2k-1},\nu_{2k})]}{\partial(\nu_{2k},\nu_{2k-1})+\partial(\nu_{2k-1},\nu_{2k})+\partial(\nu_{2k-1},\nu_{2k+1})} + \theta_4 \frac{\partial(\nu_{2k},\nu_{2k+1}).\partial(\nu_{2k-1},\nu_{2k+1})+\partial(\nu_{2k},\nu_{2k}).\partial(\nu_{2k-1},\nu_{2k+1})}{\partial(\nu_{2k},\nu_{2k+1})+\partial(\nu_{2k-1},\nu_{2k+1})+\partial(\nu_{2k-1},\nu_{2k})+\partial(\nu_{2k},\nu_{2k})}.$$

$$\leq \theta_1\partial(\nu_{2k}, \nu_{2k-1}) + (\theta_2 + \frac{\theta_4}{2})\partial(\nu_{2k}, \nu_{2k-1}).$$

Therefore,

$$\partial(\nu_{2k+1}, \nu_{2k}) \leq \Psi\partial(\nu_{2k}, \nu_{2k-1}).$$

Where $\Psi = \frac{\theta_1}{\left(1-\theta_2-\frac{\theta_4}{2}\right)} < 1$, but $2\theta_1 + 2\theta_2 + \theta_4 < 2$.

So, continuing this process, we get

$$\partial(\nu_{2k+1}, \nu_{2k}) \leq \Psi^k\partial(\nu_0, \nu_1).$$

Now for $l > k$ and using the triangle inequality, we obtain

$$\partial(\nu_{2k}, \nu_{2l}) \leq \partial(\nu_{2k}, \nu_{2k+1}) + \partial(\nu_{2k+1}, \nu_{2k+2}) + \cdots + \partial(\nu_{2l-1}, \nu_{2l})$$

$$\leq \Psi^{2k}\partial(\nu_0,\nu_1) + \Psi^{2k+1}\partial(\nu_0,\nu_1) + \cdots + \Psi^{2k+2l-1}\partial(\nu_0,\nu_1)$$
$$\leq (\Psi^{2k} + \Psi^{2k+1} + \cdots + \Psi^{2k+2l-1})\partial(\nu_0,\nu_1)$$
$$\leq \frac{\Psi^{2k}}{1-\Psi}\,\partial(\nu_0,\nu_1) \to 0.$$

Hence $\partial(\nu_{2k},\nu_{2l}) \to 0$, $k,l \to \infty$. Hence in $\mathfrak{J}$, $\{\nu_{2k}\}$ is a Cauchy sequence. Due to the fact that $(\mathfrak{J},\partial)$ is a complete cone metric spaces, $\exists\ \nu^* \in \mathfrak{J}$ Such that $\nu_k \to \nu^*$ as $k \to \infty$. Again, since $\jmath$ and $\hbar$ are continuous such that

$$\jmath(\nu) = \jmath(\lim_{k\to\infty} \nu_{2k}) = \lim_{k\to\infty} \jmath\nu_{2k} = \lim_{k\to\infty} \nu_{2k+1} = \nu^* \text{ and}$$
$$\hbar(\nu) = \jmath\hbar(\lim_{k\to\infty} \nu_{2k}) = \lim_{k\to\infty} \hbar\nu_{2k} = \lim_{k\to\infty} \nu_{2k+2} = \nu^*.$$

Hence ν^* is common fixed point of $\jmath$ and $\hbar$ in $\mathfrak{J}$.

Now we will show that ν^* is a unique common fixed point of $\jmath$ and $\hbar$ in $\mathfrak{J}$.

Suppose ξ^* is another common fixed point of $\jmath$ and $\hbar$ in$\mathfrak{J}$. Then from (7.1) we have

$$\partial(\nu^*,\xi^*) = \partial(\jmath\nu^*,\hbar\xi^*)$$
$$\leq \theta_1\partial(\nu^*,\xi^*) + \theta_2\frac{\partial(\nu^*,\ \jmath\Gamma\nu^*).\partial(\xi^*,\hbar\xi^*)+\partial(\nu^*,\hbar\xi^*).\partial(\xi^*,\ \jmath\nu^*)}{\partial(\nu^*,\xi^*)}$$
$$+\theta_3\frac{\partial(\nu^*,\ \jmath\nu^*)[.\partial(\nu^*,\jmath\nu^*)+\partial(\xi^*,\hbar\xi^*)]}{\partial(\nu^*,\xi^*)+\partial(\xi^*,\jmath\nu^*)+\partial(\xi^*,\hbar\xi^*)} + \theta_4\frac{\partial(\nu^*,\ \jmath\nu^*).\partial(\xi^*,\ \jmath\nu^*)+\partial(\xi^*,\hbar\xi^*).\partial(\nu^*,\hbar\xi^*)}{\partial(\nu^*,\ \jmath\nu^*)+\partial(\xi^*,\ \jmath\nu^*)+\partial(\xi^*,\hbar\xi^*)+\partial(\nu^*,\hbar\xi^*)}.$$
$$\leq \theta_1\partial(\nu^*,\xi^*) + \theta_2\frac{\partial(\nu^*,\nu^*).\partial(\xi^*,\xi^*)+\partial(\nu^*,\xi^*).\partial(\xi^*,\nu^*)}{\partial(\nu^*,\xi^*)}$$
$$+\theta_3\frac{\partial(\nu^*,\nu^*)[.\partial(\nu^*,\nu^*)+\partial(\xi^*,\xi^*)]}{\partial(\nu^*,\xi^*)+\partial(\xi^*,\nu^*)+\partial(\xi^*,\xi^*)} + \theta_4\frac{\partial(\nu^*,\nu^*).\partial(\xi^*,\nu^*)+\partial(\xi^*,\xi^*).\partial(\nu^*,\xi^*)}{\partial(\nu^*,\nu^*)+\partial(\xi^*,\nu^*)+\partial(\xi^*,\xi^*)+\partial(\nu^*,\xi^*)}.$$

Thus

$\partial(\nu^*,\xi^*) \leq (\theta 1 + \theta_2)\partial(\nu^*,\xi^*) \to 0$, which is a contradiction, because $2\theta_1 + 2\theta_2 + \theta_4 < 2$, So, $\partial(\nu^*,\xi^*) = 0$. Implies that $\nu^* = \xi^*$, thus,ν^* is a unique common fixed point of $\jmath$ and $\hbar$ in$\mathfrak{J}$.

Example 8: Let $\mathfrak{J}= \left\{\frac{1}{4},\frac{1}{2},1\right\}$with partial order " $\leq$ " and $\partial(\nu,\xi) = |\nu-\xi|$ be a cone metric space on $\mathfrak{J}$ .Assume that $\jmath,\hbar : \mathfrak{J} \to \mathfrak{J}$ be defined by

$$\jmath(\nu) = \begin{cases} \nu^2\ , & \text{if } \nu \in [0,\frac{1}{2}] \\ 1-\nu\ , & \text{if } \nu \in (\frac{1}{2},1] \\ \frac{\nu}{2}\ , & \text{if } \nu \in (1,\infty) \end{cases} = \hbar(\xi)$$

To show that j and k are satisfies the contractive condition (7.1) of theorem (7), for $v = \frac{1}{2}, \xi = 1, \theta_2 = \theta_3 = \theta_4 = 1$. Then we have

$$jv = \frac{1}{4}, k\xi = \frac{1}{2}, \partial(v, jv) = \partial\left(\frac{1}{2}, j\frac{1}{2}\right) = \frac{1}{4},$$
$$\partial(\xi, k\xi) = \partial(1, k1) = \frac{1}{2},$$
$$\partial(v, k\xi) = \partial\left(\frac{1}{2}, k1\right) = 0,$$
$$\partial(\xi, jv) = \partial\left(1, j\frac{1}{2}\right) = \frac{3}{4},$$
$$\partial(v, \xi) = \partial\left(\frac{1}{2}, 1\right) = \frac{1}{2} \text{ and } \partial(jv, k\xi) = \partial\left(j\frac{1}{2}, k1\right) = \frac{1}{4}.$$

Thus, from (3.1.1), we get

$$\frac{1}{4} = \partial(jv, k\xi)$$
$$\leq \frac{1}{2}\theta_1 + \frac{1}{4} + 0 + \frac{1}{8}$$
$$= \frac{1}{2}\theta_1 + \frac{3}{8}.$$

It follows that, $\theta_1 \leq \frac{1}{4}$. Hence all the conditions of Theorem 7 are satisfied and j and k has a unique fixed point $v = \frac{1}{4} = \xi$.

Corollary 9: Consider that, $(\Im, \leq)$ be a ordered set of partially and let ∂ is a metric on $(\Im, \partial)$,where $(\Im, \partial)$ is a complete metric space. Take over, $j, k: \Im \to \Im$ be a continuous self mapping such that $v, \xi \in \Im$, $v \neq \xi$ there exist $\theta_1, \theta_2 \in [0,1)$ with $\theta_1 + \theta_2 < 2$, satisfies the following condition

$$\partial(jv, k\xi) \leq \theta_1 \partial(v, \xi) + \theta_2 \frac{\partial(v, jv).\partial(\xi, k\xi) + \partial(v, k\xi).\partial(\xi, jv)}{\partial(v, \xi)} \qquad (9.1)$$

For $v_0 \in \Im$, suppose $\{v_k\}_{k=0}^{\infty} \subset \Im$ and defined by $v_{2k+1} = jv_{2k}$ and $v_{2k+2} = kv_{2k+1}$ $k = 0,1,2 \ldots$. be the Picard iteration association to j and k. Then there exist $v_k \to v^*, \lim_{k\to\infty} j^k v = v^* = \lim_{k\to\infty} \xi$. Then v^* is a unique common fixed point.

Theorem 10: Consider that, $(\Im, \leq)$ be a ordered set of partially and let ∂ is a metric on $(\Im, \partial)$,where $(\Im, \partial)$ is a complete metric space. Take over, $\mathcal{L}_1, \mathcal{L}_2: \Im \to \Im$ be a continuous self mapping such that $q, r, \in \Im$, $q \neq r$ there exist $i_1, i_2, \in [0,1)$ with $2i_1 + i_2 < 1$, satisfies the following condition

$$\partial(\mathcal{L}_1 q, \mathcal{L}_2 r) \leq i_1 \max \left\{ \begin{array}{l} \frac{\partial(q,\mathcal{L}_1 q).\partial(r,\mathcal{L}_2 r)+\partial(q,\mathcal{L}_2 r).\partial(r,\ \mathcal{L}_1 q)}{\partial(q,\ r)}, \\ \frac{\partial(q,\ \mathcal{L}_1 q).\partial(q,\ \mathcal{L}_2 r)+\partial(r,\mathcal{L}_2 r).\partial(r,\ \mathcal{L}_1 q)}{\partial(q,\ r)} \end{array} \right\}$$

$$. \qquad + i_2 \partial(q, r)$$

For $q_0 \in \mathfrak{J}$, suppose $\{q_k\}_{k=0}^{\infty} \subset \mathfrak{J}$ and defined by $q_{2k+1} = \mathcal{L}_1 q_{2k}$ *and* $q_{2k+2} = \mathcal{L}_2 q_{2k+1}$ $k = 0,1,2$ .... be the Picard iteration association to $\mathcal{L}_1$ and $\mathcal{L}_2$. Then there exist $q_k \to q^*$, $\lim_{k\to\infty} \mathcal{L}_1{}^k q = q^* = \lim_{k\to\infty} \mathcal{L}_2 r$. Then q^* is a unique common fixed point.

Proof: : Let $q_0 \in \mathfrak{J}$, we define the iterative sequence$\{q_{2k}\}$ in $\mathfrak{J}$ defined as

$$q_{2k+1} = \mathcal{L}_1 q_{2k} = \mathcal{L}_1{}^{2k+1} q_0 ,$$

and

$$q_{2k+2} = \mathcal{L}_2 q_{2k+1} = \mathcal{L}_2{}^{2k+2} q_0, \text{ for every } k \geq 1.$$

Now put $q = q_{2k}$ and $r = q_{2k-1}$ in (9.1) we have

$$\partial(q_{2k+1}, q_{2k}) = \partial(\mathcal{L}_1 q_{2k}, \mathcal{L}_2 q_{2k-1})$$

$$\leq i_1 max \left\{ \begin{array}{l} \frac{\partial(q_{2k},\ \mathcal{L}_1 q_{2k}).\partial(q_{2k-1},\mathcal{L}_2 q_{2k-1})+\partial(q_{2k},\ \mathcal{L}_2 q_{2k-1}).\partial(q_{2k-1},\ \mathcal{L}_1 q_{2k})}{\partial(q_{2k},q_{2k-1})} \\ \frac{\partial(q_{2k},\mathcal{L}_1 q_{2k}).\partial(q_{2k},\mathcal{L}_2 q_{2k-1})+\partial(q_{2k-1},\ \mathcal{L}_2 q_{2k-1}).\partial(q_{2k},\ \mathcal{L}_1 q_{2k})}{\partial(q_{2k},q_{2k-1})} \end{array} \right\}$$

$$+ i_2 \partial(q_{2k}, q_{2k-1}).$$

$$\leq i_1 max \left\{ \begin{array}{l} \frac{\partial(q_{2k},q_{2k+1}).\partial(q_{2k-1},q_{2k})+\partial(q_{2k},q_{2k}).\partial(q_{2k-1},q_{2k+1})}{\partial(q_{2k},q_{2k-1})} \\ \frac{\partial(q_{2k},q_{2k+1}).\partial(q_{2k},\ q_{2k})+\partial(q_{2k-1},q_{2k}).\partial(q_{2k},q_{2k+1})}{\partial(q_{2k},q_{2k-1})} \end{array} \right\}$$

$$+\ i_2 \partial(v_{2k}, v_{2k-1}).$$

$$\leq i_1 max\{\partial(q_{2k}, q_{2k+1}), \partial(q_{2k}, q_{2k+1})\} + i_2 \partial(v_{2k}, v_{2k-1}).$$

$$= i_1 \partial(q_{2k}, q_{2k+1}) + i_2 \partial(v_{2k}, v_{2k-1}).$$

Thus

$\partial(q_{2k+1}, q_{2k}) \leq \frac{i_2}{1-i_1} \partial(v_{2k}, v_{2k-1})$. Since $w = \frac{i_2}{1-i_1} < 1$.

Therefore, $\partial(q_{2k+1}, q_{2k}) \leq w\, \partial(v_{2k}, v_{2k-1})$. Continuing this process, we get

$$\partial(q_{2k+1}, q_{2k}) \leq w^k \partial(q_0, q_1).$$

Now for $l > k$ and using the triangle inequality, we obtain

$$\partial(q_{2k}, q_{2l}) \leq \frac{w^{2k}}{1-w} \partial(q_0, q_1) \to 0.$$

Hence $\partial(q_{2k}, q_{2l}) \to 0$, $k, l \to \infty$. Hence in $\mathfrak{J}$, $\{q_{2k}\}$ is a Cauchy sequence. Due to the fact that $(\mathfrak{J}, \partial)$ is a complete cone metric spaces, $\exists\ q^* \in \mathfrak{J}$ Such that $q_{2k} \to q^*$ as $k \to \infty$. Again, since $\mathcal{L}_1$ and $\mathcal{L}_2$ are continuous such that

$$\mathcal{L}_1(q) = \mathcal{L}_1(\lim_{k\to\infty} q_{2k}) = \lim_{k\to\infty} \mathcal{L}_1 q_{2k} = \lim_{k\to\infty} q_{2k+1} = q^* \text{ and}$$

$$\mathcal{L}_2(q) = \mathcal{L}_2(\lim_{k\to\infty} q) = \lim_{k\to\infty} \mathcal{L}_2 q_{2k} = \lim_{k\to\infty} q_{2k+2} = q^*.$$

Since $\mathcal{L}_1(q) = q^* = \mathcal{L}_2(q)$. Therefore, q^* is common fixed point of $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathfrak{J}$.

Now we will show that q^* is a unique common fixed point of $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathfrak{J}$.

Suppose r^* is another common fixed point of $\mathcal{L}_1$ and $\mathcal{L}_2$ in$\mathfrak{J}$. Then from (9.1) we have

$$\partial(q^*, r^*) = \partial(\mathcal{L}_1 q^*, \mathcal{L}_2 r^*)$$

$$\leq i_1 \max \left\{ \begin{matrix} \frac{\partial(q^*, \mathcal{L}_1 q^*).\partial(r^*, \mathcal{L}_2 r^*) + \partial(q^*, \mathcal{L}_2 r^*).\partial(r^*,\ \mathcal{L}_1 q^*)}{\partial(q^*, r^*)}, \\ \frac{\partial(q^*, \mathcal{L}_1 q^*).\partial(q^*, \mathcal{L}_2 r^*) + \partial(r^*, \mathcal{L}_2 r^*).\partial(r^*,\ \mathcal{L}_1 q^*)}{\partial(q,\ r)} \end{matrix} \right\} + i_2 \partial(q^*, r^*)$$

$$\leq (i_1 + i_2)\, \partial(q^*, r^*)$$

$\partial(q^*, r^*) \leq (i_1 + i_2)\, \partial(q^*, r^*) \to 0$, which is a contradiction, because $2i_1 + i_2 < 1$.

So, $\partial(q^*, r^*) = 0$. Implies that $q^* = r^*$, thus,q^* is a unique common fixed point of $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathfrak{J}$. This completes the proof of the theorem.

Lemma 11[Jovanovic, M. et al. (2010): Let $\{q_k\}$ be a sequence in a metric type space $(\mathfrak{J}, d)$ such that

$$d(q_k, q_{k+1}) \leq \sigma d(q_{k-1}, q_k)$$

for some $o < \sigma < 1$, and each $k = 1,2,3 \ldots \ldots$, then $\{q_k\}$ is a Cauchy sequence in$(\mathfrak{J}, d)$.

Proposition 12: Presume that $(\mathfrak{J}, d)$ is a metric space and $\mathcal{T}: \mathfrak{J} \to \mathfrak{J}$ be a given function and $\{q_{2k}\}$ be Picard sequence of initial point $q_0 \in \mathfrak{J}$ satisfying the following condition:

$d(q_{2k}, q_{2k+1}) \leq \frac{d(q_{2k-1}, q_{2k}) + d(q_{2k-1}, q_{2k+1}) + d(q_{2k}, q_{2k+1}) + d(q_{2k}, q_{2k+2}) + n}{d(q_{2k-1}, q_{2k}) + d(q_{2k-1}, q_{2k+1}) + d(q_{2k}, q_{2k+1}) + d(q_{2k}, q_{2k+2}) + n}\ d(q_{2k-1}, q_{2k})$ …. (12.1) where $m, n \in \mathbb{R}^+$ such that $\mathfrak{n} < m$. Then $\{q_k\}$ is a Cauchy sequence.

Proof: Suppose that $q_0 \in \mathfrak{J}$ be an arbitrary point and let $\{q_{2k}\}$ be sequence of initial point q_0 . If $q_{2k_0} = q_{2k_0-1}$ for some $q_0 \in \mathbb{N}$, then q_{k_0} is a fixed point of $\mathcal{T}$ and so $\{q_{2k}\}$ is a Cauchy sequence. Let us suppose that (12.1) satisfied for the sequence $\{q_{2k}\}$. If $q_{2k_0} \neq q_{2k_0-1}$ for some $k \in \mathbb{N}$, from (12.1) we deduce that the sequence $\{d(q_{2k}, q_{2k+1})\}$ is decreasing. Thus there exist a non negative real number r such

that $d(q_{2k}, q_{2k+1}) \to r$. Then, we claim that $r = 0$. If $r > 0$, on taking limit as $k \to \infty$ on both side of (10.1), we get

$r \leq \frac{r+\cdots+r+n}{r+\cdots+r+m}\, r < r$, which is a contradiction. It follows that $r = 0$.

Now show that $\{q_{2k}\}$ is a Cauchy sequence .Suppose $\sigma \in [0,1)$. Since $r = 0$, there exist$k(\sigma) \in \mathbb{N}$ such that

$$d(q_{2k}, q_{2k+1}) \leq \frac{d(q_{2k-1}, q_{2k}) + d(q_{2k-1}, q_{2k+1}) + d(q_{2k}, q_{2k+1}) + d(q_{2k}, q_{2k+2}) + n}{d(q_{2k-1}, q_{2k}) + d(q_{2k-1}, q_{2k+1}) + d(q_{2k}, q_{2k+1}) + d(q_{2k}, q_{2k+2}) + n} \leq \sigma.$$

For all $k \geq k(\sigma)$.

This implies that $d(q_{2k}, q_{2k+1}) \leq d(q_{2k-1}, q_{2k})$ for all $k \geq k(\sigma)$ and by Lemma 11 $\{q_{2k}\}$ is a Cauchy sequence. The proof that$\{q_{2k}\}$ is a Cauchy sequence if (10.1) holds is the same.

Theorem 13: Consider that, $(\mathfrak{J}, \leq)$ be a ordered set of partially and let ∂ is a metric on $(\mathfrak{J}, \partial)$,where $(\mathfrak{J}, \partial)$ is a complete metric space. Take over, $\mathcal{L}_1, \mathcal{L}_2 : \mathfrak{J} \to \mathfrak{J}$ be a continuous self mapping such that $q, r, \in \mathfrak{J}$, $q \neq r$ there exist $\alpha \in [0,1)$ satisfies the following condition

$$d(\mathcal{L}_1 q, \mathcal{L}_2 r) \leq \frac{d(q,\mathcal{L}_2 r)+d(q,\mathcal{L}_2(\mathcal{L}_2 r)+d(r,\mathcal{L}_1 q)+d(r,\mathcal{L}_1(\mathcal{L}_1 q)+n}{d(q,\mathcal{L}_1 q)+d(q,\mathcal{L}_1(\mathcal{L}_1 q)+d(r,\mathcal{L}_2 r)+d(r,\mathcal{L}_2(\mathcal{L}_2 r)+m}\, d(q, r) + \alpha d(r, \mathcal{L}_1 q)\ldots \quad (13.1)$$

For $q_0 \in \mathfrak{J}$, suppose $\{q_{2k}\}_{k=0}^{\infty} \subset \mathfrak{J}$ and defined by $q_{2k+1} = \mathcal{L}_1 q_{2k}$ and $q_{2k+2} = \mathcal{L}_2 q_{2k+1}$ $k = 0,1,2$ .... be the Picard iteration association to $\mathcal{L}_1$ and $\mathcal{L}_2$. Then there exist $q_{2k} \to q^*$, $\lim_{k\to\infty} \mathcal{L}_1{}^k q = q^* = \lim_{k\to\infty} r$. Then q^* is a unique common fixed point.

Proof: Let $q_0 \in \mathfrak{J}$, we define the iterative sequence $\{q_{2k}\}$ in $\mathfrak{J}$ defined as

$$q_{2k+1} = \mathcal{L}_1 q_{2k} = \mathcal{L}_1{}^{2k+1} q_0 ,$$

and

$$q_{2k+2} = \mathcal{L}_2 q_{2k+1} = \mathcal{L}_2{}^{2k+2} q_0, \text{ for every } k \geq 1.$$

Now put $q = q_{2k-1}$ and $r = q_{2k}$ in (11.1) we have

$$d(q_{2k}, q_{2k+1}) = d(\mathcal{L}_1 q_{2k-1}, \mathcal{L}_2 q_{2k}) \leq \frac{d(q_{2k-1},\mathcal{L}_2 q_{2k})+d(q_{2k-1},\mathcal{L}_2(\mathcal{L}_2 q_{2k})+d(q_{2k},\mathcal{L}_1 q_{2k-1})+d(q_{2k},\mathcal{L}_1(\mathcal{L}_1 q_{2k-1})+n}{d(q,\mathcal{L}_1 q_{2k-1})+d(q,\mathcal{L}_1(\mathcal{L}_1 q_{2k-1})+d(q_{2k},\mathcal{L}_2 q_{2k})+d(q_{2k},\mathcal{L}_2(\mathcal{L}_2 q_{2k})+m}\, d(q_{2k-1}, q_{2k}) + \alpha d(q_{2k}, \mathcal{L}_1 q_{2k-1})$$

$$\leq \frac{d(q_{2k-1},q_{2k+1})+d(q_{2k-1},\mathcal{L}_2(q_{2k+1})+d(q_{2k},\ q_{2k})+d(q_{2k},\mathcal{L}_1(q_{2k})+n}{d(q_{2k-1},\ q_{2k})+d(q_{2k-1},\mathcal{L}_1(q_{2k})+d(q_{2k},q_{2k+1})+d(q_{2k},\mathcal{L}_2(q_{2k+1})+m}\ d(q_{2k-1},q_{2k})$$

$$+\alpha d(q_{2k},q_{2k})$$

$$\leq \frac{d(q_{2k-1},q_{2k+1})+d(q_{2k-1},q_{2k+2})+d(q_{2k},q_{2k+1})+n}{d(q_{2k-1},\ q_{2k})+d(q_{2k-1},q_{2k+1})+d(q_{2k},q_{2k+1})+d(q_{2k},q_{2k+2})+m}\ d(q_{2k-1},q_{2k})$$

$$\leq \frac{[d(q_{2k-1},q_{2k})+d(q_{2k},q_{2k+2})]+d(q_{2k-1},q_{2k+1})+d(q_{2k},q_{2k+1})+n}{d(q_{2k-1},\ q_{2k})+d(q_{2k-1},q_{2k+1})+d(q_{2k},q_{2k+1})+d(q_{2k},q_{2k+2})+m}\ d(q_{2k-1},q_{2k}).$$

That is, condition (13.1) holds for the sequence $\{q_{2k}\}$.Then, by proposition (12.1), $\{q_{2k}\}$ is a Cauchy sequence. Since $\mathfrak{J}$ is a complete metric space, the sequence$\{q_{2k}\}$ converges to some $x \in \mathfrak{J}$. Now, we prove that x is a common fixed point of $\mathcal{L}_1 and\ \mathcal{L}_2$. Using (13.1) with $q = q_{2k}$ and $r = x$, then we get

$$d(q_{2k+1},\mathcal{L}_2x) = d(\mathcal{L}_1q_{2k},\mathcal{L}_2x)$$

$$\leq \left[\frac{d(q_{2k},\mathcal{L}_2x)+d(q_{2k},\mathcal{L}_2(\mathcal{L}_2x)+\ d(x,\mathcal{L}_1q_{2k})+\ d(x,\mathcal{L}_1(\mathcal{L}_1q_{2k})+n}{d(q_{2k},\mathcal{L}_1\ q_{2k})+d(q_{2k},\mathcal{L}_1(\mathcal{L}_1q_{2k})+d(x,\mathcal{L}_2x)+\ d(x,\mathcal{L}_2\ (\mathcal{L}_2x)+m}\right]\ d(q_{2k},x)$$

$$+\ \alpha d(x,\mathcal{L}_1q_{2k}$$

$$\leq \left[\frac{d(q_{2k},\mathcal{L}_2x)+d(q_{2k},\mathcal{L}_2(\mathcal{L}_2x)+\ d(x,q_{2k+1})+d(x,q_{2k+2})+n}{d(q_{2k},q_{2k+1})+d(q_{2k},\ q_{2k+2})+d(x,\mathcal{L}_2x)+d(x,\mathcal{L}_2\ (\mathcal{L}_2x)+m}\right]\ d(q_{2k},x)$$

$$+\ \alpha d(x,q_{2k+1})$$

On taking limit as $k \to \infty$ on both sides, we get

$$d(x,\mathcal{L}_2x) \leq \left[\frac{d(x,\mathcal{L}_2x)+d(q_{2k},\mathcal{L}_2(\mathcal{L}_2x)+n}{d(x,\mathcal{L}_2x)+d(x,\mathcal{L}_2\ (\mathcal{L}_2x)+m}\right]\ \lim_{n\to\infty} d(q_{2k},x)$$

$$+\ \lim_{n\to\infty} \alpha d(x,q_{2k+1}) = 0.$$

This implies that $d(x,\mathcal{L}_2x) = 0 \Rightarrow x = \mathcal{L}_2x$. Similarly, we can show that $x = \mathcal{L}_1x$. Since $\mathcal{L}_1x = x = \mathcal{L}_2x$. Therefore, x is common fixed point of $\mathcal{L}_1$and $\mathcal{L}_2$.

Now we will show that x is a unique common fixed point of $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathfrak{J}$.

Suppose y is another common fixed point of $\mathcal{L}_1$and $\mathcal{L}_2$ in$\mathfrak{J}$. Then from (13.1) with $q = x\ and\ r = y$ we get

$$d(x,y) = d(\mathcal{L}_1x,\mathcal{L}_2y)$$

$$\leq \left[\frac{d(x,\mathcal{L}_2y)+\ d(y,\mathcal{L}_1x)+d(x,\mathcal{L}_2(\mathcal{L}_2y)+\ d(y,\mathcal{L}_1(\mathcal{L}_1x)+n}{d(x,\mathcal{L}_1x)+d(x,\mathcal{L}_1(\mathcal{L}_1x)+d(y,\mathcal{L}_2y)+d(y,\mathcal{L}_2(\mathcal{L}_2y)+m}\right]\ d(x,y) + \alpha d(y,\mathcal{L}_1x)$$

$$= \frac{4\ d(x,y)+n}{m} + \alpha d(y,x) \to 0.$$

Therefore, $d(x,y) = 0 \Rightarrow x = y$. Thus x is unique common fixed point of $\mathcal{L}_1$and $\mathcal{L}_2$ in$\mathfrak{J}$. This completes the proof of the theorem.

If $n = m = 0$ in Theorem 13, we get the following Corollary

Corollary 14: Consider that, $(\mathfrak{J}, \leq)$ be a ordered set of partially and let ∂ is a metric on $(\mathfrak{J}, \partial)$,where $(\mathfrak{J}, \partial)$ is a complete metric space. Take over, $\mathcal{L}_1, \mathcal{L}_2 : \mathfrak{J} \to \mathfrak{J}$ be a continuous self mapping such that $q, r, \in \mathfrak{J}$, $q \neq r$ there exist $\alpha \in [0,1)$ satisfies the following condition

$$d(\mathcal{L}_1 q, \mathcal{L}_2 r) \leq \frac{d(q,\mathcal{L}_2 r)+d(q,\mathcal{L}_2(\mathcal{L}_2 r)+d(r,\mathcal{L}_1 q)+d(r,\mathcal{L}_1(\mathcal{L}_1 q)}{d(q,\mathcal{L}_1 q)+d(q,\mathcal{L}_1(\mathcal{L}_1 q)+d(r,\mathcal{L}_2 r)+d(r,\mathcal{L}_2(\mathcal{L}_2 r)} \, d(q, r) + \alpha d(r, \mathcal{L}_1 q) \ldots$$

For $q_0 \in \mathfrak{J}$, suppose $\{q_{2k}\}_{k=0}^{\infty} \subset \mathfrak{J}$ and defined by $q_{2k+1} = \mathcal{L}_1 q_{2k}$ and $q_{2k+2} = \mathcal{L}_2 q_{2k+1}$ $k = 0,1,2$ .... be the Picard iteration association to $\mathcal{L}_1$ and $\mathcal{L}_2$. Then there exist $q_{2k} \to q^*, \lim_{k\to\infty} \mathcal{L}_1{}^k q = q^* = \lim_{k\to\infty} r$. Then q^* is a unique common fixed point.

If $n = m = 1$ in Theorem 13, we get the following Corollary

Corollary 15: Consider that, $(\mathfrak{J}, \leq)$ be a ordered set of partially and let ∂ is a metric on $(\mathfrak{J}, \partial)$,where $(\mathfrak{J}, \partial)$ is a complete metric space. Take over, $\mathcal{L}_1, \mathcal{L}_2 : \mathfrak{J} \to \mathfrak{J}$ be a continuous self mapping such that $q, r, \in \mathfrak{J}$, $q \neq r$ there exist $\alpha \in [0,1)$ satisfies the following condition

$$d(\mathcal{L}_1 q, \mathcal{L}_2 r) \leq \frac{d(q,\mathcal{L}_2 r)+d(q,\mathcal{L}_2(\mathcal{L}_2 r)+d(r,\mathcal{L}_1 q)+d(r,\mathcal{L}_1(\mathcal{L}_1 q)+1}{d(q,\mathcal{L}_1 q)+d(q,\mathcal{L}_1(\mathcal{L}_1 q)+d(r,\mathcal{L}_2 r)+d(r,\mathcal{L}_2(\mathcal{L}_2 r)+1} \, d(q, r) + \alpha d(r, \mathcal{L}_1 q) \ldots$$

For $q_0 \in \mathfrak{J}$, suppose $\{q_{2k}\}_{k=0}^{\infty} \subset \mathfrak{J}$ and defined by $q_{2k+1} = \mathcal{L}_1 q_{2k}$ and $q_{2k+2} = \mathcal{L}_2 q_{2k+1}$ $k = 0,1,2$ .... be the Picard iteration association to $\mathcal{L}_1$ and $\mathcal{L}_2$. Then there exist $q_{2k} \to q^*, \lim_{k\to\infty} \mathcal{L}_1{}^k q = q^* = \lim_{k\to\infty} r$. Then q^* is a unique common fixed point.

CONCLUSION

We have investigated and extended the current literature of Bhardwaj R. et al. (2007), Chouhan et al. (2014), and Garg and Priyanka (2016) and obtained unique common fixed point theorems for rational type contraction in metric spaces. Theorem 5 with Example 6, Theorem 7 with Example 8 and Corollary 9, Theorem 10 with Proposition 12, and Theorem 13 with Corollaries 14 and 15 are our results.

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