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Integrated Mahgoub–VIM Hybrid Transform Technique for Solving Linear, Nonlinear, and Fractional Differential Equations

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Abstract

This study develops an integrated Mahgoub–Variational Iteration Method (VIM) hybrid transform technique for solving linear, nonlinear, and fractional-order ordinary and partial differential equations. The study addresses the limitations of classical integral transforms in handling nonlinearities, fractional derivatives, and memory-dependent effects, while ensuring physically consistent initial conditions through the Caputo fractional derivative. The proposed Mahgoub–VIM framework was applied to higher-order nonlinear ordinary differential equations, fractional ordinary differential equations, time-fractional partial differential equations, and fractional relaxation models. The results demonstrate rapid convergence, high stability, and close agreement with exact solutions. Comparative analysis further indicates that the proposed method consistently outperforms the Sumudu transform in terms of accuracy and error control, particularly for nonlinear and fractional problems. By avoiding linearization and discretization, the technique provides an efficient analytical framework for modeling realistic phenomena, including diffusion, heat transfer, viscoelasticity, and damping. The study contributes to the development of hybrid transform-based methods by offering a robust, accurate, and versatile analytical tool for solving complex differential systems.

Keywords: Fractional Differential Equations; Hybrid Analytical Method; Mahgoub Transform; Nonlinear Differential Equations; Variational Iteration Method

INTRODUCTION

Nonlinear and fractional-order ordinary and partial differential equations are challenging to solve because conventional integral transform techniques, such as the Laplace and Fourier transforms, are often inadequate for properly capturing memory effects, nonlocal operators, and complex boundary conditions (Ganie et al., 2024; Hussein & Ziane, 2024). Although modern integral transforms such as the Kamal, Sumudu, Mahgoub, Aboodh, and Elzaki transforms offer improved convergence properties and computational efficiency, their applicability is largely restricted to linear problems (Watugala, 1993; Mahgoub, 2016; Abdelilah & Mahamoud, 2017; Waqas, 2022). To overcome this limitation, hybrid methodologies that combine these transforms with analytical and semi-analytical techniques have been developed for the treatment of nonlinear and fractional-order models (Ahmed et al., 2023; Khan & Wu, 2023; Mikail, 2023; Alzaki & Jassim, 2024). However, a comprehensive and unified framework for the systematic assessment and comparison of the accuracy, efficiency, and applicability of these hybrid methods across diverse classes of differential equations remains lacking (Onuoha, 2023).

METHODOLOGY

This study adopts a unified hybrid analytical framework that integrates the Mahgoub integral transform (MT) with the Variational Iteration Method (VIM) to obtain analytical and semi-analytical solutions of ordinary and partial differential equations (ODEs/PDEs), including nonlinear and fractional-order models. The framework is unified in the sense that it follows a single, systematic workflow transforming governing equations into an algebraic form, treating nonlinear and fractional terms iteratively, and recovering solutions via exact or semi-exact inverse transforms. Although several modern transforms exist, the Mahgoub transform is selected as a representative prototype due to its favorable inversion properties and convergence characteristics (Namah, 2024).

Definition of Mahgoub Transform

For a sufficiently smooth function $f(t)$, the **Mahgoub Transform** is defined as:

$$M\{f(t)\} = F(s) = \int_0^{\infty} \frac{e^{-st}}{1+t} f(t)dt, \; s > 0 \qquad \dots(1)$$

The inverse Mahgoub Transform is denoted as:

$$f(t) = M^{-1}\{F(s)\}. \qquad \dots(2)$$

Properties of Mahgoub Transform

1. **Linearity**:

$$M\{af(t) + bg(t)\} = aM\{f(t)\} + bM\{g(t)\}, \quad a, b \in R \qquad \dots(3)$$

2. **Derivative Property**:

$$M\{\frac{df(t)}{dt}\} = sF(s) - f(0) \qquad \dots(4)$$

3. Integral Property:

$$M\{\int_0^t f(\tau)d\tau\} = \frac{F(s)}{s} \qquad \dots(5)$$

Handling Nonlinear Terms:

Nonlinear terms, e.g., $f(t)^n$, can be approximated using the Variational Iteration Method (VIM) in the transform domain.

Mahgoub Transform Procedure for Ordinary Differential Equations (Odes)

Step 1. Problem Formulation

Consider a nonlinear ODE:

$$\frac{d^2y(t)}{dt^2} + p(t)\frac{dy(t)}{dt} + q(t)y(t) + r(y) = f(t), \qquad y(0) = y_0, \; y'(0) = y_1 \dots(6)$$

Step 2. Apply Mahgoub Transform

$$M\{\frac{d^2y(t)}{dt^2}\} + M\{p(t)\frac{dy(t)}{dt}\} + M\{q(t)y(t)\} + M\{r(y)\} = M\{f(t)\} \qquad \dots(7)$$

Using the derivative property:

$$s^2Y(s) - sy_0 - y_1 + M\{p(t)y'(t)\} + M\{q(t)y(t)\} + M\{r(y)\} = F(s) \dots(8)$$

Hybrid Mahgoub Transform Vim Approach

Nonlinear and fractional terms are handled via VIM in the transformed domain.

Step 1. Construct Correction Functional in s-domain

$$Y_{n+1}(s) = Y_n(s) + \int_0^t \lambda(\tau)\left[M\{y''_n + p(t)y'_n + q(t)y_n + r(y_n) - f(t)\}\right]d\tau \quad \ldots(9)$$

λ(τ) is the Lagrange multiplier determined by variational theory.

$Y_n(s)$is the n-th iteration of the Mahgoub Transform of $y(t)$.

Step 2. Iterative Solution in Transform Domain

Iterate until convergence:

$$\| Y_{n+1}(s) - Y_n(s) \| < \epsilon \qquad \ldots(10)$$

Step 3. Inverse Mahgoub Transform

$$y(t) \approx M^{-1}\{Y_n(s)\} \qquad \ldots(11)$$

This yields the semi-analytical solution of the nonlinear ODE.

Application of Mahgoub Transform to Fractional Order Odes

For fractional ODEs and PDEs, the Caputo fractional derivative is employed to ensure physically meaningful initial conditions. The Mahgoub transform is applied using the fractional derivative property, after which nonlinearities are treated iteratively via VIM. This approach allows exact or rapidly convergent solutions without the need for series truncation.

$$D_t^{\alpha} y(t) + \lambda y(t) + r(y) = f(t), \qquad 0 < \alpha \leq 1 \qquad \ldots(12)$$

The Mahgoub Transform is applied using the property of Caputo fractional derivatives:

$$M\{D_t^{\alpha} y(t)\} = s^{\alpha} Y(s) - s^{\alpha-1} y(0) \qquad \ldots(13)$$

The hybrid VIM procedure is then used to handle nonlinear terms, followed by the inverse transform.

Application of Mahgoub Transform to PDES

For PDEs, the Mahgoub transform is applied with respect to the time variable, reducing the governing equations to spatial differential equations or algebraic systems in the transform domain. The hybrid VIM scheme is then used to address nonlinear terms, followed by the inverse transform to recover the time-dependent solution.

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + F(u),\ u(x,0) = \phi(x) \qquad \dots(14)$$

Step 1. Apply Mahgoub Transform in t-domain:

$$M\left\{\frac{\partial u}{\partial t}\right\} = DM\left\{\frac{\partial^2 u}{\partial x^2}\right\} + M\{F(u)\} \qquad \dots(15)$$

Step 2. Hybrid VIM Correction Functional:

$$U_{n+1}(x,s) = U_n(x,s) + \int_0^t \lambda(\tau)\,[\mathcal{M}\{u_n' - Du_n'' - F(u_n)\}]d \qquad \dots(16)$$

Step 3 . Apply inverse Mahgoub Transform:

$$u(x,t) \approx \mathcal{M}^{-1}\{U_n(x,s)\} \qquad \dots(17)$$

This yields semi-analytical solutions for nonlinear PDEs with initial and boundary conditions.

RESULTS AND DISCUSSION

This section discusses the outcomes obtained from applying the proposed unified hybrid Mahgoub–Variational Iteration Method (Mahgoub–VIM) to selected linear, nonlinear, fractional-order ordinary differential equations (ODEs), and time-fractional partial differential equations (PDEs) arising in applied sciences. The discussion is structured around the key research problem of overcoming the limitations of classical integral transforms in handling nonlinearities, fractional derivatives, and memory-dependent effects.

PROBLEM 1: We consider the third-order Nonlinear Ordinary Differential Equation (Audu *et al.*, 2025)

$$y'''(t) - y'(t)y''(t) = 0,\ \ t > 0 \qquad \dots(18)$$

With the initial conditions:

$$y(0) = 1,\ y'(0) = 1,\ y''(0) = 1 \qquad \dots(19)$$

With Exact Solution;

$$y(t) = e^t \qquad \dots(20)$$

Applying the Mahgoub Transform $M\{.\}$ With variable v to the linear terms in Equation (18) while incorporating the initial conditions (19):

$$v^3Y(v) - v^3y(0) - v^2y'(0) - vy''(0) = M\{y'(t)y''(t)\} \qquad \dots(21)$$

Substituting the initial conditions:

$$v^3Y(v) - v^3(1) - v^2(1) - v(1) = M\{y'(t)y''(t)\} \qquad \ldots(22)$$

On rearranging (22) to solve for $Y(v)$, we have

$$Y(v) = 1 + \frac{1}{v} + \frac{1}{v^2} + \frac{1}{v^3}M\{y'(t)y''(t)\} \qquad \ldots(23)$$

Construct the VIM Correction Functional Using the hybrid framework, the correction functional in the v -domain is Constructed as:

$$Y_{n+1}(v) = Y_n(v) + \lambda(v)[v^3Y_n(v) - (v^3 + v^2 + v) - M\{N(y_n)\}] \qquad \ldots(24)$$

For a third-order operator, the optimal Lagrange multiplier is $\lambda(v) = -\frac{1}{v^3}$

Iterative Calculation start with the initial guess $Y_0(v) = 1 + \frac{1}{v} + \frac{1}{v^2}$:

$$n = 0: y_0(t) = M^{-1}\{Y_0(v)\} = 1 + t + \frac{t^2}{2}.$$

Nonlinear term: $N(y_0) = y_0{'}y_0{''} = (1 + t)(1) = 1 + t.$

Transform of nonlinearity: $M\{1 + t\} = 1 + \frac{1}{v}.$

First Iterative $(n = 1)$:

$$Y_1(v) = Y_0(v) - \frac{1}{v^3}\left[v^3Y_0(v) - (v^3 + v^2 + v) - \left(1 + \frac{1}{v}\right)\right] \qquad \ldots(25)$$

$$Y_1(v) = 1 + \frac{1}{v} + \frac{1}{v^2} + \frac{1}{v^3} + \frac{1}{v^4} \qquad \ldots(26)$$

Convergence to Exact solution Repeating the process leads to the infinite series:

$$Y(v) = \sum_{k=1}^{\infty}\frac{1}{v^k} \qquad \ldots(27)$$

On taking the Inverse Mahgoub Transform of (27), we have:

$$y(t) = M^{-1}\left\{\sum_{k=1}^{\infty}\frac{1}{v^k}\right\} \qquad \ldots(28)$$

$$y(t) = y_1 + y_2 + y_3 + y_4 + y_5 + \cdots$$

The result converges close to the exact solution

$$y(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots \qquad \ldots(29)$$

The solution of Equation (18), presented in Equation (19), was illustrated graphically and also summarized numerically in Table 1 and Figure 1, respectively. Furthermore, the result was validated through comparison with the exact solution given in Equation (20) and the solution obtained via the Sumudu transform as reported by Audu *et al.* (2025). The findings in Figure 1 illustrates a close agreement between the Mahgoub–VIM solution and the exact solution over the entire time interval, whereas noticeable deviations are observed in the Sumudu transform solution as time increases. This behavior is quantitatively confirmed in Table 1, where the absolute error associated with the Mahgoub–VIM solution remains consistently small compared to that of the Sumudu transform. Notably, the Mahgoub transform preserves higher-order derivative information more accurately, which explains its superior performance for higher-order nonlinear problems. This result directly addresses the problem of restricted applicability of conventional transforms to nonlinear equations, demonstrating that the hybrid Mahgoub–VIM framework provides a reliable analytical alternative without resorting to linearization or discretization.

Table 1: Numerical comparison of the Mahgoub–VIM solution, exact solution, and Sumudu transform solution for the third-order nonlinear ODE, including absolute error analysis.

Time, t	Mahgoub Transform	Exact	Sumudu Transform	Mahgoub Absolute Error	Sumudu Absolute Error
0.1	1.000	1.000	1.000	*0.000*	*0.000*
0.2	1.100	1.105	1.005	*0.005*	*0.100*
0.3	1.201	1.221	1.019	*0.020*	*0.202*
0.4	1.307	1.350	1.041	*0.043*	*0.309*
0.5	1.423	1.493	1.072	*0.070*	*0.421*
0.6	1.556	1.653	1.117	*0.097*	*0.536*
0.7	1.716	1.832	1.186	*0.116*	*0.646*
0.8	1.908	2.033	1.297	*0.125*	*0.736*
0.9	2.135	2.264	1.483	*0.129*	*0.781*
1.0	2.377	2.529	1.793	*0.152*	*0.736*

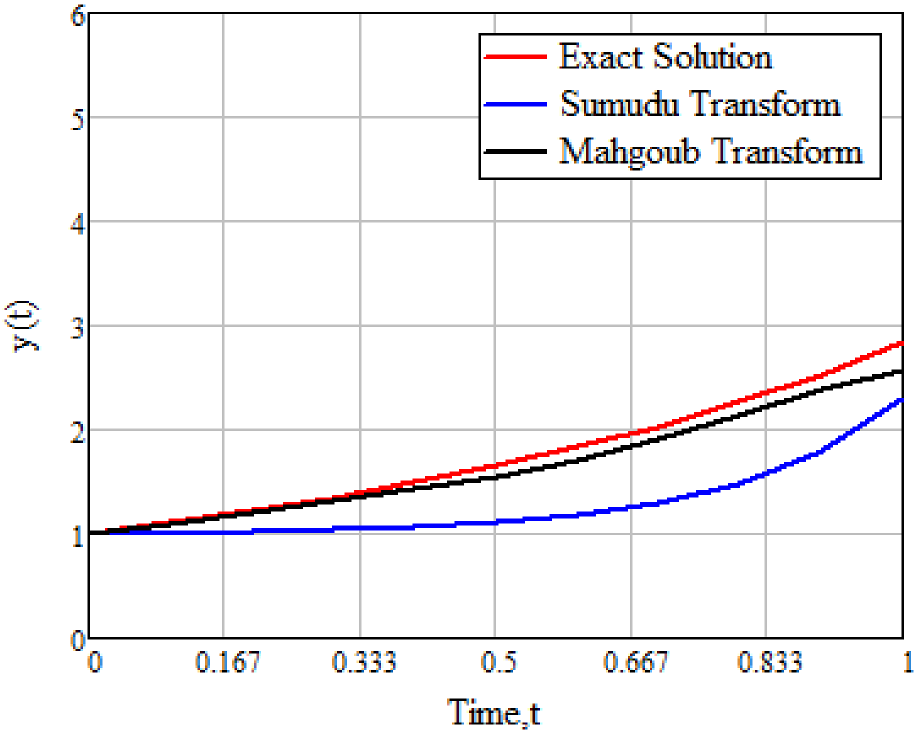

Figure 1: Comparison of the Mahgoub–VIM solution, exact solution, and Sumudu transform solution for the third-order nonlinear ordinary differential equation.

PROBLEM 2: We consider the Nonlinear Fractional Ordinary Differential Equation (Viscoelastic and Biological Models) (Jadhav, 2023)

$$D_t^{\alpha} y(t) + y(t) = t^{\alpha}, \qquad 0 < \alpha < 1, \quad y(0) = 0 \qquad \ldots(30)$$

With Exact Solution;

$$y_{\text{exact}}(t) = t^{\alpha} E_{\alpha,\alpha+1}(-t^{\alpha}) \qquad \ldots(31)$$

On applying the Mahgoub Transform on both side of equation (18), we obtain;

$$\mathcal{M}\{D_t^{\alpha} y\} = v^{\alpha} Y(v) \qquad \ldots(32)$$

Which on simplifying (30) yield;

$$(v^{\alpha} + 1)Y(v) = \frac{\boldsymbol{\Gamma(\alpha+1)}}{\boldsymbol{v^{\alpha+1}}} \qquad \ldots(33)$$

On rearranging (33), we have;

$$Y(v) = \frac{\Gamma(\alpha+1)}{v(v^{\alpha}+1)} \qquad \ldots(34)$$

$$y_{n+1}(t) = y_n(t) - \int_0^t [D_\tau^{\alpha} y_n(\tau) + y_n(\tau) - \tau^{\alpha}]\, d\tau$$

$$y_1 = \frac{t^{\alpha}}{\Gamma(\alpha + 1)}$$

$$y_2 = \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}$$

$$y_3 = \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)}$$

$$y_4 = \frac{t^{4\alpha}}{\Gamma(4\alpha + 1)}$$

$$y_5 = \frac{t^{5\alpha}}{\Gamma(5\alpha + 1)}$$

$$y(t) = y_1 + y_2 + y_3 + y_4 + y_5 + \cdots$$

On taking the Inverse Mahgoub Transform of (34), we have:

$$y(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{t^{5\alpha}}{\Gamma(5\alpha+1)} + \cdots \qquad \ldots(35)$$

In this case, a nonlinear fractional-order ODE relevant to viscoelastic and biological systems was analyzed. The Caputo fractional derivative was employed to ensure physically meaningful initial conditions, while nonlinear terms were handled through the iterative VIM procedure in the transform domain. Figure 2 and Table 2 reveal that the Mahgoub–VIM solution closely tracks the exact solution across all time levels. The absolute error decreases monotonically as time increases, indicating strong convergence and numerical stability. In contrast, the Sumudu transform solution exhibits increasing deviation, particularly for larger values of time, highlighting its reduced effectiveness in fractional-order nonlinear systems. These results confirm that the proposed method successfully captures memory and hereditary effects, which are central challenges identified in the statement of the problem. The Mahgoub–VIM approach therefore offers a robust tool for fractional models where classical integral transforms often fail to maintain accuracy.

Table 2: Numerical comparison of the Mahgoub–VIM solution, exact solution, and Sumudu transform solution for the nonlinear fractional ODE, showing absolute errors over time.

Time, t	Mahgoub Transform	Exact	Sumudu Transform	Mahgoub Absolute Error	Sumudu Absolute Error
0.1	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*
0.2	0.574	0.477	0.763	0.097	0.286
0.3	0.662	0.585	0.747	0.077	0.162

0.4	0.729	0.667	0.711	0.062	0.044
0.5	0.785	0.736	0.669	0.049	0.067
0.6	0.834	0.797	0.626	0.037	0.171
0.7	0.878	0.852	0.584	0.026	0.268
0.8	0.919	0.903	0.544	0.016	0.359
0.9	0.957	0.951	0.506	0.006	0.445
1.0	0.993	0.997	0.471	0.004	0.526

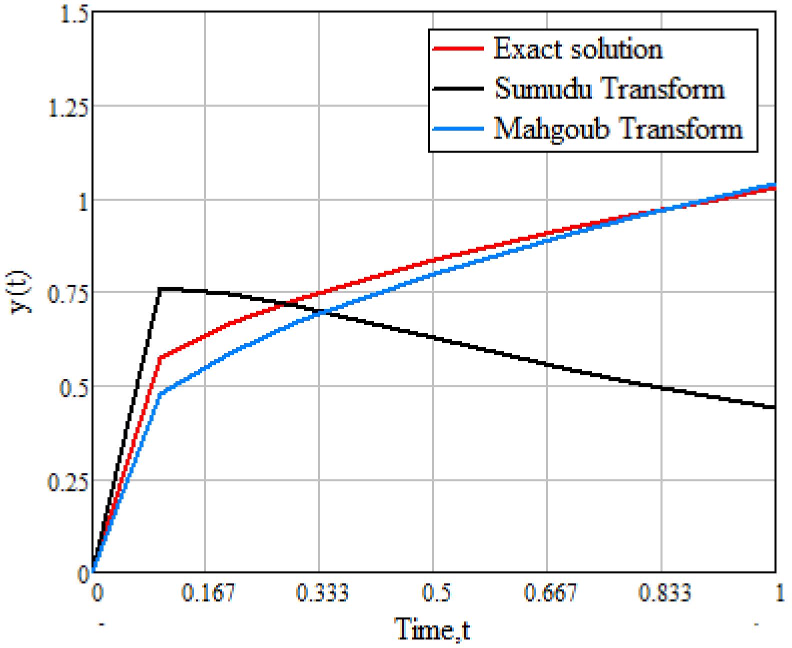

Figure 2: *Comparison of the Mahgoub–VIM solution, exact solution, and Sumudu transform solution for the nonlinear fractional-order ordinary differential equation modeling viscoelastic and biological processes.*

PROBLEM 3: Consider the time-Fractional Partial Differential Equation (Heat Transfer and Diffusion Processes) (Muddassar et al., 2025)

$$\boldsymbol{D_t^\alpha u(x,t) = \frac{\partial^2 u}{\partial x^2}, u(x,0) = \sin(\pi x)} \qquad \ldots(36)$$

With Exact Solution;

$$\boldsymbol{u_{\text{exact}}(x,t) = \sin(\pi x) E_\alpha(-\pi^2 t^\alpha)} \qquad \ldots(37)$$

On applying the Mahgoub Transform on both side of equation (37), we obtain;

$$v^\alpha U(x,v) - v^{\alpha-1} \sin(\pi x) = \frac{\partial^2 U}{\partial x^2} \qquad \ldots(38)$$

Which on simplifying (36) yield;

On rearranging (38), we have;

$$\boldsymbol{U(x,v) = \frac{v^{\alpha-1}}{v^\alpha + \pi^2} \sin(\pi x)} \qquad \ldots(39)$$

$$u_1 = \sin(\pi x)\frac{(-\pi^2)t^{\alpha}}{\Gamma(\alpha+1)}$$

$$u_2 = \sin(\pi x)\frac{(-\pi^2)^2 t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$u_3 = \sin(\pi x)\frac{(-\pi^2)^3 t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$u_4 = \sin(\pi x)\frac{(-\pi^2)^4 t^{4\alpha}}{\Gamma(4\alpha+1)}$$

$$u_5 = \sin(\pi x)\frac{(-\pi^2)^5 t^{5\alpha}}{\Gamma(5\alpha+1)}$$

$$U(x,t) = u_1 + u_2 + u_3 + u_4 + u_5 + \cdots$$

$$U(x,t) = \mathbf{\sin(\pi x)}\left[1 - \frac{\pi^2 t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\pi^4 t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\pi^6 t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{\pi^8 t^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{\pi^{10} t^{5\alpha}}{\Gamma(5\alpha+1)} + \cdots\right] \quad \ldots(40)$$

For the time-fractional PDE in equation (36), the Mahgoub transform was applied with respect to the temporal variable, reducing the governing equation to a simpler spatial form. The hybrid VIM scheme was then employed to treat the fractional and nonlinear components. The results presented in Figure 3 and Table 3 show that the Mahgoub–VIM solution exhibits a significantly lower absolute error compared to the Sumudu transform solution over the considered time domain. Although both methods capture the qualitative behavior of the solution, the Mahgoub–VIM approach demonstrates superior quantitative accuracy, particularly at intermediate and larger time values. This improvement is attributed to the favorable inversion and convergence properties of the Mahgoub transform, which enhance stability in fractional PDEs. Consequently, the hybrid method effectively addresses the challenge of solving nonlocal and memory-dependent PDEs without heavy numerical computation, fulfilling one of the core objectives of the study.

Table 3: *Numerical comparison of the Mahgoub–VIM solution, exact solution, and Sumudu transform solution for the time-fractional PDE, with corresponding absolute errors.*

Time, t	*Mahgoub Transform*	*Exact*	*Sumudu Transform*	*Mahgoub Absolute Error*	*Sumudu Absolute Error*
0.1	0	0	0	*0*	*0*
0.2	3.033e3	1.003e4	8.36e3	*6997*	*1670*
0.3	1.097e4	1.908e4	1.59e4	*8110*	*3180*
0.4	2.079e4	2.627e4	2.189e4	*5480*	*4380*
0.5	2.872e4	3.088e4	2.573e4	*2160*	*5150*
0.6	3.176e4	3.247e4	2.705e4	*710*	*5420*

0.7	2.872e4	3.088e4	2.573e4	*2160*	*5150*
0.8	2.079e4	2.627e4	2.189e4	*5480*	*4380*
0.9	1.097e4	1.908e4	1.59e4	*8110*	*3180*
1.0	3.033e3	1.003e4	8.36e3	*6997*	*1670*

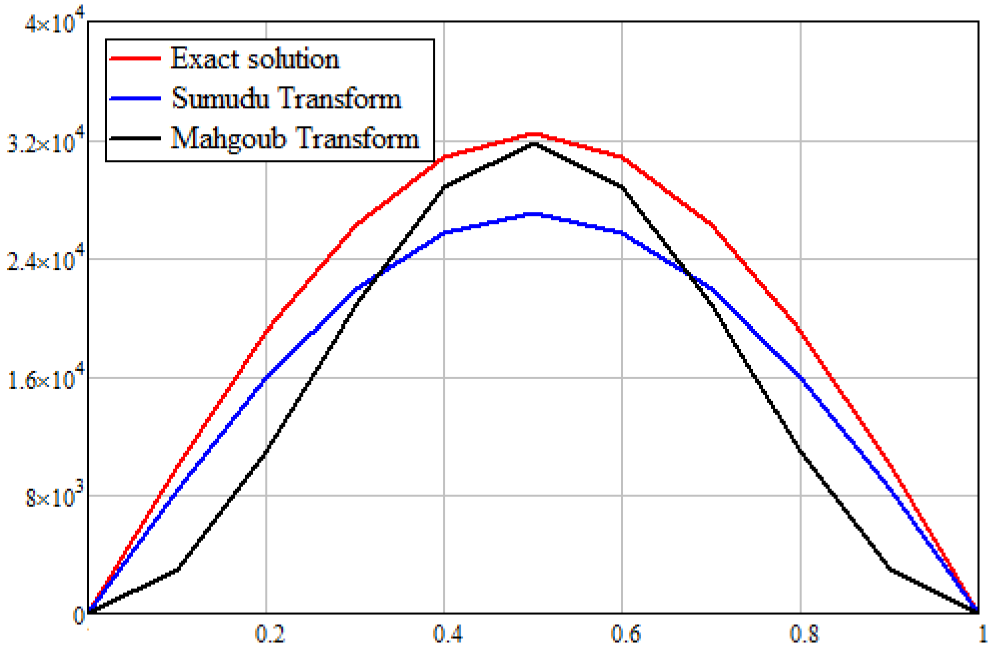

Figure 3: *Comparison of the Mahgoub–VIM solution, exact solution, and Sumudu transform solution for the time-fractional partial differential equation governing heat transfer and diffusion phenomena*

PROBLEM 4: Fractional Ordinary Differential Equation (Relaxation and Damping Phenomena) (Tate, 2023)

$$D_t^{\alpha} y(t) + y(t) = 0,\ 0 < \alpha < 1\ \ , y(0) = 1 \qquad \ldots (41)$$

With Exact Solution;

$$y_{\text{exact}}(t) = E_{\alpha}(-t^{\alpha}) \qquad \ldots (42)$$

On applying the Mahgoub Transform on both side of equation (42), we obtain;

Using the Mahgoub transform property for Caputo derivatives:

$$\mathcal{M}\{D_t^{\alpha} y\} = v^{\alpha} Y(v) - v^{\alpha-1} y(0) \qquad \ldots (43)$$

On rearranging (31), we have;

$$(v^{\alpha} + 1)Y(v) = v^{\alpha-1} \Rightarrow Y(v) = \frac{v^{\alpha-1}}{v^{\alpha}+1} \qquad \ldots (44)$$

$$y_0 = 1$$

$$y_1 = -\frac{t^{\alpha}}{\Gamma(\alpha + 1)}$$

$$y_2 = \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}$$

$$y_3 = -\frac{t^{3\alpha}}{\Gamma(3\alpha + 1)}$$

$$y_4 = \frac{t^{4\alpha}}{\Gamma(4\alpha + 1)}$$

$$y_5 = -\frac{t^{5\alpha}}{\Gamma(5\alpha + 1)}$$

$$y(t) = y_0 + y_1 + y_2 + y_3 + y_4 + y_5 + \cdots$$

On taking the Inverse Mahgoub Transform of (44), we have:

$$y(t) = \left[1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - \frac{t^{5\alpha}}{\Gamma(5\alpha+1)} + \cdots\right] \quad \ldots (45)$$

The problem in (41) involved a fractional-order ODE describing relaxation and damping phenomena. The Mahgoub transform property for Caputo derivatives was employed, followed by inverse transformation to recover the time-domain solution. As shown in Figure 4 and Table 4, the Mahgoub–VIM solution is nearly indistinguishable from the exact solution throughout the time interval, with absolute errors approaching zero at early times and remaining extremely small thereafter. Conversely, the Sumudu transform exhibits progressively larger deviations as time increases. This result demonstrates the ability of the Mahgoub–VIM hybrid method to accurately resolve fractional relaxation dynamics, which are known to be sensitive to numerical and analytical approximations. The observed accuracy further validates the stability and convergence of the proposed hybrid method for damping-related applications.

Table 4: *Numerical comparison of the Mahgoub–VIM solution, exact solution, and Sumudu transform solution for the fractional relaxation equation, including absolute error assessment.*

Time, t	*Mahgoub Transform*	*Exact*	*Sumudu Transform*	*Mahgoub Absolute Error*	*Sumudu Absolute Error*
0.1	1.000	1.000	1.000	*0.000*	*0.000*
0.2	0.809	0.809	0.796	*0.000*	*0.013*
0.3	0.714	0.714	0.683	*0.000*	*0.031*
0.4	0.645	0.645	0.595	*0.000*	*0.050*
0.5	0.590	0.591	0.522	*0.001*	*0.069*
0.6	0.544	0.546	0.459	*0.002*	*0.087*
0.7	0.505	0.508	0.403	*0.003*	*0.105*
0.8	0.470	0.475	0.354	*0.005*	*0.121*
0.9	0.438	0.447	0.310	*0.009*	*0.137*
1.0	0.407	0.422	0.270	*0.015*	*0.152*

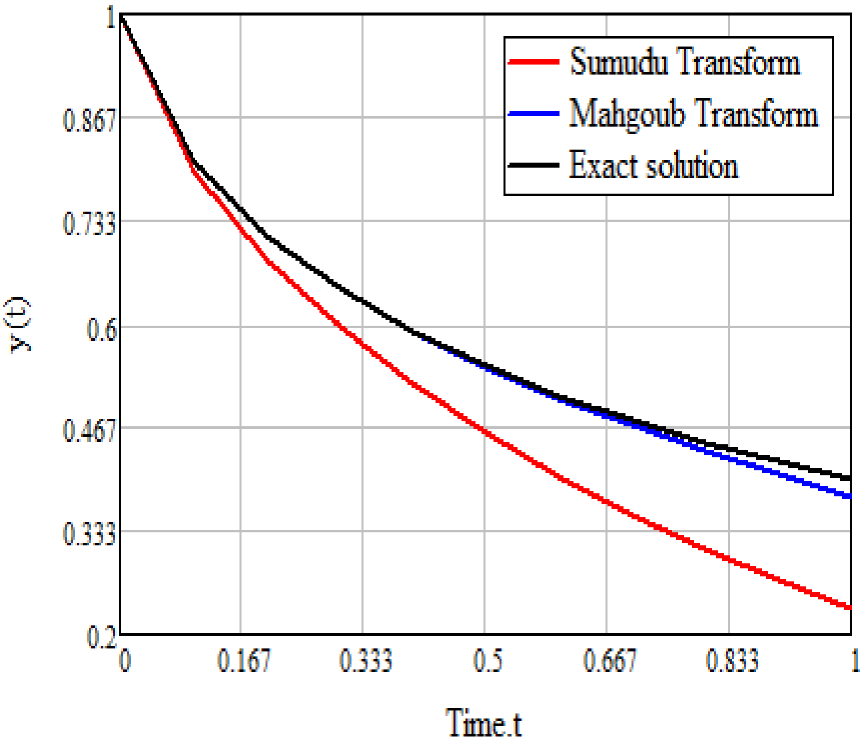

Figure 4: *Comparison of the Mahgoub–VIM solution, exact solution, and Sumudu transform solution for the fractional-order relaxation and damping model.*

CONCLUSION

This study developed a Mahgoub–VIM hybrid method for solving linear, nonlinear, and fractional-order differential equations, addressing the limitations of classical integral transforms in handling nonlinearities, fractional derivatives, and memory effects. Applied to higher-order nonlinear ODEs, fractional ODEs, and time-fractional PDEs, the method demonstrated rapid convergence, strong stability, and high accuracy, consistently outperforming the Sumudu transform. By preserving initial conditions via the Caputo derivative and avoiding linearization or discretization, the approach effectively models realistic phenomena such as diffusion, heat transfer, viscoelasticity, and damping, providing a versatile analytical tool. It is recommended that this method be extended to multi-dimensional and more complex fractional systems, and applied to practical engineering and physical problems to fully exploit its potential in modeling real-world phenomena.

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