Asian Journal of
Science, Technology, Engineering, and Art

e-ISSN: 3025-4507
p-ISSN: 3025-5287

Index: Harvard, Boston, Sydney University, Dimensions, Lens, Scilit, Semantic, Google, etc

https://doi.org/10.58578/AJSTEA.v4i3.9126

Convergence Theorems for Total Asymptotically Nonexpansive Mappings in CAT (0) Spaces

Ibrahim Usman Garba, M.I Bello, M.S Adamu, Pofi Ephraim
Kurgwi Qua'an Pan L. G. C. Plateau State, Nigeria
isahabdullahi7474@gmail.com

Article Info:

Submitted:	Revised:	Accepted:	Published:
Feb 2, 2026	Apr 30, 2026	May 12, 2026	May 17, 2026

Abstract

This paper proposes new iterative algorithms for total asymptotically nonexpansive mappings in CAT(0) spaces. The study aims to establish a strong convergence theorem for the proposed algorithms under suitable mathematical conditions. Using a theoretical analytical approach, the convergence properties of the iterative schemes are examined within the geometric framework of CAT(0) spaces. The results demonstrate that the proposed algorithms converge strongly to a fixed point of total asymptotically nonexpansive mappings under the stated assumptions. These findings improve and extend several recent results reported in the literature on nonlinear mappings and fixed point theory. The study contributes to the advancement of convergence theory in CAT(0) spaces by providing refined iterative methods and strengthening the theoretical foundation for analyzing total asymptotically nonexpansive mappings.

Keywords: CAT(0) Spaces; Fixed Point Theory; Iterative Algorithms; Strong Convergence; Total Asymptotically Nonexpansive Mappings

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Introduction

The study of metric spaces without linear structure has played a vital role in various branches of pure and applied sciences one of such spaces is the CAT(0) spaces. The initials of the term CAT are in honor of Cartan , Alexandrov and Toponogov, who have made remarkable contributions toward the understanding of curvature via inequalities for distance function. In 1957 Alexandrov gave several definitions of what it means for a metric space to have curvature k. Let M_k^2 denote the following spaces; $if\ k < 0$ then M_k^2 is a real hyperbolic space H^2 with metric scaled by $\frac{1}{\sqrt{-k}}$; if $k = 0$ then M_k^2 is the Euclidean space E^n; if $k > 0$ then M_k^2 is the 2-sphere S^2 with the metric scaled by a factor$\frac{1}{\sqrt{k}}$. Alexadrov pointed out that one can define curvature bounds on a space by comparing triangles in that space to triangle in M_k^2.

The "0" in "CAT(0)" refers to the fact that the comparison space $\mathbb{R}^2$ has a curvature 0. More generally, there is a notion of CAT(k) space for any real number k and that the real number k stands for curvature. What makes CAT(0) of interest among researchers is the following; first it covers many important spaces such as the hyperbolic spaces. Second, if $k > 0$, the geodesic segment in the sphere joining two points say $x\ and\ y$ will only exist if the points are closer together (that is, if for any two points we take the distance between them, $d(x, y) < \frac{2\pi}{\sqrt{k}}$), which is a restriction and so studying CAT(k) for $k > 0$ is not of interest to many researchers. For K < 0 , the spaces are covered by CAT(0) from the following established result; if X is a CAT(k) space, then it is CAT(k') for every $k \leq k'$. (Bridson and Haefliger 1999). Therefore, CAT(0) space turns out to be of interest to many researchers.

Many authors have made a lot of efforts to generalize the fixed point theory from Euclidean spaces to CAT (0) spaces. Fixed point theory in CAT (0) spaces was first studied by Kirk (2003 ,2004). He showed that every nonexpansive (single-valued) mapping defined on bounded closed convex subset of a complete CAT (0) space always has a fixed point. Since then, the fixed-point theory for single-valued and multivalued mapping in CAT (0) spaces has been rapidly developed, and many papers have appeared Chaoha,P. and Phon-on,A. (2006),. Dhompongsa, et al (2005), Shahzad N.(2009).

Let (X, d) be a metric space. A geodesic path joining $x \in X\ to\ y \in X$ (or more briefly, a geodesic from $x\ to\ y$) is a map c from a closed interval $[0, L] \subset R\ to\ X$

$such\ that\ c(0) = x, c(l) = y,$ and $d(c(t_1), c(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in [0, l]$. In particular, c is an isometry and $d(x, y) = l.$ The image $\propto of\ c$ is called a geodesic (or metric) segment $joining\ x\ and\ y.$ When it is unique, this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be $geodesic\ space$ if every two points of X are joined by a geodesic, and X is said to be $uniquely\ geodesic$ if there is exactly one geodesic joining $x\ and\ y\ for\ x, y \in X.$ A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.

A geodesic triangle represented by $\Delta(x, y, z)$, in a geodesic metric space (X, d) consists of three points $x, y\ and\ z\ in\ X$ (the vertices of Δ) and a geodesic segments between each pair of vertices (the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x, y, z)\ in\ (X, d)$ is a triangle $\bar{\Delta}(x, y, z) := \Delta(\bar{x}, \bar{y}, \bar{z})$, in the Euclidean plane $\mathbb{R}^2$ such that $d(x, y) = d_{\mathbb{R}^2}(\bar{x}, \bar{y})$

$d(x, z) = d_{\mathbb{R}^2}(\bar{x}, \bar{z})$ and $d(y, z) = d_{\mathbb{R}^2}(\bar{y}, \bar{z})$. A geodesic space is said to be a CAT (0) space if all geodesic triangles satisfy the following comparison axiom: Let Δ be a geodesic triangle in X and let $\bar{\Delta}$ be its comparison triangle in $\mathbb{R}^2$. Then Δ is said to satisfy CAT (0) inequality, if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$

$$\boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y}) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}). \tag{1.1}$$

If x, y_1, y_2 are points in CAT(0) space and if y_0 is the midpoint of the segment $[\![y_1, y_2]\!]$ which we will denote by $(y_1 \oplus y_2)/2$, then the CAT(0) inequality implies

$$d^2(x, \tfrac{y_1 \oplus y_2}{2}) \leq \tfrac{1}{2} d^2(x, y_1) + \tfrac{1}{2} d^2(x, y_2) - \tfrac{1}{4} d(y_1, y_2) \tag{1.2}$$

Inequality (1.2) is known as the (CN) inequality of Bruhat and Titz . In fact, a geodesic space is a CAT(0) space if and only if it satisfies the CN inequality. This is obtainable using the triangle inequality, and is unique up to isometry on $\mathbb{R}^2$. Bridson and Haefliger (1999) have shown that such a triangle always exists. The above inequality (1.2) has been extended in Dhompongsa et al 2008 as

(i) $d\big((1 - \alpha)x \oplus \alpha y, z\big) \leq (1 - \alpha) d(x, z) + \alpha d(y, z)$ for all $\alpha \in [0,1]\ and\ x, y, z \in X.$

(ii) let X be a CAT(0) space then

$$d((1 - \alpha)x \oplus \alpha y, z)^2 \leq (1 - \alpha) d(x, z)^2 + \alpha d(y, z)^2 - \alpha(1 - \alpha) d(x, y)^2$$

for any $\alpha \in [0,1]\ and\ x, y, z \in X.$

Examples of CAT(0) spaces : Euclidean buildings (Bacâk 2014, Brown K. S. 1989), Hilbert spaces(Bridson, M and Haefliger 1999)- the only Bannach spaces which are CAT(0), ℝ-Trees(Kirk, W.A 2004) : a metric T is an ℝ-Trees if

i) $for\ x, y \epsilon T\ there\ is\ unique\ geodesic\ [x, y]$

ii) $if\ [x, y] \cap [y, x] = \{y\}, then\ [x, z] = [x, y] \cup [y, z]$

Classical hyperbolic spaces $\mathcal{H}^n$ (Goebel and Reich,1984),Complete simply connected Riemannian manifolds with nonpositive sectional curvature.(Rose Moris 2018). Complete CAT(0) spaces are often called Hadamard spaces.

Definition 1.0

Let C be a nonempty subset a CAT (0) space and $T : C \rightarrow C$ a mapping. A point $x \in C$ is a fixed point of $T\ if\ Tx = x.$ Denote the set of all fixed points of $T\ by\ F(T).$ We say that the mapping T is

(i) contractive if there exists $k \in (0,1)\ such\ that\ d(Tx, Ty) \leq kd(x, y) for\ all\ x, y \in C$

(ii) nonexpansive $if\ d(Tx, Ty) \leq d(x, y)\ for\ all\ x, y \in C$

(iii) asymptotically nonexpansive if there exist a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $d(T^n x, T^n y) \leq k_n d(x, y) for\ all\ x, y \in C,\ n \geq 1$

(iv) generalized asymptotically nonexpansive if there are nonnegative sequences $\{k_n^1\}\ and\ \{k_n^2\}\ with\ k_n^1 \rightarrow 0\ and\ k_n^2 \rightarrow 0\ such\ that\ d(T^n x, T^n y) \leq (1 + k_n^1)\ d(x, y) + k_n^2$ for all $x, y \in C, n \geq 1.$

(v) asymptotically nonexpansive in the intermediate sense it is continuous and lim $sup_{n\to\infty}$ $sup_{x,y \in C}(d(T^n x, T^n y) \leq 0$

(vi) Albert et al, 2006. First introduced the concept of total asymptotically nonexpansive mappings. T is said to be (μ, γ, ψ)-total asymptotically nonexpansive, If there exist non-negative real sequences $\{\mu_n\}, \{\gamma_n\}\ with\ \mu_n \rightarrow 0, \gamma_n \rightarrow 0$ as $n \rightarrow \infty$ and a continuous strictly increasing function $\psi : [0, \infty) \rightarrow [0, \infty)\ with\ \psi(0) = 0$ such that

$$d(T^n x, T^n y) \leq d(x, y) + \mu_n \psi(d(x, y)) + \gamma_n \text{ for all } x, y \in C, n \geq 1.$$

(vii) uniformly $L - Lipschitzian\ if\ d(T^n x, T^n y) \leq L\ d(x, y)\ for\ all\ x, y \in C. n \geq 1$

(viii) uniformly continuous if for each $\epsilon > 0, there\ exist\ \delta(\epsilon) > 0\ such\ that\ d(Tx, Ty) < \varepsilon$ whenever $d(x, y) < \delta$.

Remark

Every uniformly L-Lipschitzian mapping is uniformly continuous but the converse is not true in general. The function $T(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$ but not Lipschitz. The class of total asymptotically nonexpansive mappings is the most general as it includes the classes of mapping mention in (ii) – (VI)

A point $x \in C$ is called a fixed point of $T\ if\ x = T(x)$. We denote with $F(T)$ the fixed point of T. A sequence $\{x_n\}\ in\ C$ is said to be $approaximating\ fixed\ point\ sequence$ (AFPS) of T if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

The class of such mapping generalized the basic concepts of asymptotically nonexpansive mappings introduced by kirk and Goebel 1972. Likewise, the basics concept of nearly asymptotically nonexpansive was introduced by Sahu 2005.

Relationship between the above definition.

$$\begin{array}{ccccc} Nonexpansive & \Rightarrow & Asymptotically\ nonexpansive & \Rightarrow & Total\ asymptotically\ nonexpansive \\ \Downarrow & & \Downarrow & & \Downarrow \end{array}$$

Quasi-nonexpansive ⇒asymptotically quasi-nonexpansive ⇒Total asymptotically quasi-nonexpansive

Iterative Methods

We present some notable and important iterative schemes for approximating the fixed point of nonlinear mappings.

Iterative approximation of fixed points of total nonexpansive mappings has also been studied by Chidume and Ofoedu 2007, Albet, Y et al 2008 , Mukhamedov and Saburov 2010 Qin, X. et al 2011.

In 2013 Khan introduced a new iteration process for nonexpansive mappings,which he called 'Picard-Mann hybrid iteration process' for approximating a fixed point of a nonexpansive mapping T in Bannach spaces. The sequence $\{x_n\}$ in this process is given by

$$\begin{cases} x_1 \in C \\ y_n = (1 - \alpha_n)x_n + \alpha_n T x_n \\ x_{n+1} = T y_n \end{cases} \tag{2.1.1}$$

$\forall n \geq 1, where \{\alpha_n\}$ is an appropriate sequence in the interval (0,1). He claimed that his process is independent of Picard and Mann iteration process and the convergence process is faster than Picard and Mann iteration process.

Thakur et al (2015) presented modified hybrid Picard-Mann iteration process $\{u_n\}$ – to approximate the fixed points of total asymptotically nonexpansive mappings in the frame work of CAT(0) space and the sequence $\{u_n\}$ is defined as follows:

$$u_1 \in C,$$

$$v_n = (1 - \alpha_n)u_n \oplus \alpha_n T^n u_n$$

$$u_{n+1} = T^n v_n \qquad (2 .1.2)$$

$\forall n \geq 1, where \{\alpha_n\}$ is an appropriate sequence in the interval (0,1). They also proved its convergence analysis under some certain conditions

Pansuwan and Sintunavarat 2016 introduced the Picard-Ishikawa hybrid iteration process $\{u_n\}$ which is given by

$$u_1 \in C,$$

$$\begin{cases} w_n = (1 - \alpha_n)u_n \oplus \alpha_n T^n u_n \\ v_n = (1 - \beta_n)w_n \oplus \beta_n T^n w_n \\ x_{n+1} = T^n v_n \end{cases} \qquad (2.1.3)$$

$\forall n \geq 1$ $where$ $\{\alpha_n\} and \{\beta_n\}$ are real appropriate sequences in the interval (0,1), they prove strong and $\Delta -$ convergence theorem of iteration process 2.1.3 for total asymptotically nonexpansive in complete CAT(0) spaces and they also gave numerical example.

In 2020 Kuman et al presented modified Picard –S hybrid iteration process $\{u_n\}$ as follows:

$$u_1 \in C.$$

$$w_n = (1 - \alpha_n)u_n \oplus \alpha_n T^n u_n$$

$$v_n = (1 - \beta_n)T^n u_n \oplus \beta_n T^n w_n$$

$$u_{n+1} = T^n v_n \qquad (2. 1.4\}$$

$\forall\, n \geq 1$ $where$ $\{\alpha_n\} and$ $\{\beta_n\}$ are appropriate sequences in the interval (0,1) and they established some convergence theorems to approximate the fixed points of total asymptotically nonexpansive in the setting of CAT(0) spaces.

Motivated and inspired by the researches going on in this direction especially inspired by Pim and Paiwan. (2020) and Kuman et al (2020) we will introduced a new iterative scheme, which is defined as follows

$$u_i \in C$$

$$w_n = (1-\alpha_n)u_n \oplus \alpha_n T^n u_n$$

$$V_n = T^n[(1-\beta_n)T^n u_n \oplus \beta_n T^n w_n] \tag{2.1.5}$$

$$u_{n+1} = T^n V_n$$

$for\ all\ n \geq 1, where$ $\{\alpha_n\}$ $and\{\beta_n\}$ are appropriate sequences in the interval $(0,1)$. We will prove some convergence theorems of the sequence generated by iterative scheme (2.1.5) to approximate the fixed point of total asymptotically nonexpansive mapping in CAT(0) space.

preliminaries and lemmas

The existence of fixed points for asymptotically nonexpansive mapping in CAT(0) spaces was first proved by Kirk 2004. Karapinar et al 2014 proved the existence theorem of fixed points demiclosedness principle for uniformly continuous and total asymptotically nonexpansive mappings in CAT(0) spaces as the following statement

Lemma 1.1.1 Let C be a nonempty bounded closed and convex subset of a complete CAT(0) space X and $T: C \to C$ be total asymptotically nonexpansive. Then T has a fixed point, and the fixed point set $F(T)$is closed and convex.

Let C be a nonempty closed convex subset of a CAT (0) space X, and let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we set $r(x,\{x_n\}) = \lim\sup_{n\to\infty} d(x_n, x)$.

The asymptotic radius of $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(,\{x_n\}) = inf\{r(x,\{x_n\}): x \in X\},$$

And the asymptotic center $A\ (\{x_n\})$ of $\{x_n\}$ is the set

$$A\ (\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known that in a complete CAT(0) space, $A\ (\{x_n\})$ consists of exactly one point Dhompongsa et al (2006). Also, every CAT(0) space has the $Opial\ property,$ i.e. if $\{u_n\}$ is a sequence in K, and $\Delta - \lim_{n\to\infty} u_n = x$, and then for each $y \neq x\ \in K$

$$\lim_{n\to\infty} sup\, d(u_n, x) < \lim_{n\to\infty} sup\, d(u_n, y)$$

The concept of $\Delta -$ convergence in a fundamental metric space was reported by Lim 1975. Kirk and Panyanak 2008 used the notion $\Delta -$ convergence begin by Lim 1975 to prove on the CAT(0) space analogous of some Banach space results,which relate to weak convergence. Dhompongsa and Panyanak 2008 achieved $\Delta -$ convergence theorems for Picard,Mann and Ishikawa iterations in CAT(0) space

Definition.1.2 kirk W.A.(2003). A sequence $\{x_n\}$ in metric space X is said to $\Delta - converge\ to\ x \in X\ if\ x$ is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}\ of\ \{x_n\}$. In this case we write $\Delta - \lim_{n\to\infty} x_n = x$ and $x\ is\ called\ the\ \Delta - limit\ of\ \{x_n\}$.

The following assertions in a CAT(0) space holds:

1. **Lemma 1.1.2** Kirk and Panyanak (2008). Every bounded sequence in a complete CAT(0) space always has a $\Delta -$ convergent subsequence

2. **Lemma 1.1.3** Dhompongsa et al (2007). If C is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C then the asymptotic center of $\{x_n\}$ is in C.

3. **Lemma 1.1.4 D**hompongsa and Panyanak (2008).Assume that (X, d) is a complete CAT(0) space. Let $\{u_n\}$ be a bounded sequence in X.if $A\ (\{u_n\})= \{p\}$, $\{w_n\}$ is a subsequence of $\{u_n\}\ such\ that\ A(\{w_n\}) = \{w\}$ and $\{d(x_n, u)\}$ converges, then $p = w$.

Lemma 1.1.5 Karapinar et al (2014).Assume (X, d) is a complete CAT(0) space and C is a closed,convex subset of X. Define $T : K \to K$ is a uniformly continuous and total asymptotically nonexpansive mapping . For every bounded sequence $\{u_n\} \subset K$ such that ,$lim_{n\to\infty} d(u_n, Tu_n) = 0$ and $\Delta - lim_{n\to\infty} u_n = q$. Then $Tq = q$

Lemma 1.1.6 Laowang and Panyanak (2010). Let (X,d) be a complete CAT(0) space and $u \in X$ Suppose $\{t_n\}$ is a sequence in $[b,c]$ for some $b,c \in (0,1)$ *and* $\{u_n\},\{v_n\}$ are sequences in X such that $\lim_{n\to\infty} sup\, d(u_n,u) \le r,\ \lim_{n\to\infty} sup\, d(v_n,u) \le r,$ and $\lim_{n\to\infty} d\,(t_n v_n \oplus (1-t_n)u_n, u) = r$ hold for some $r \ge 0$ then $\lim_{n\to\infty} d\,(u_n, v_n) = 0.$

Lemma 1.1.7 Dhomponsa and Panyanak (2008). Let (X,d) be a CAT(0) space.

(i) For $x,y \in X$ *and* $t \in [0,1]$, there exist a unique point $z \in [x,y]$ such that

$$d(x,z) = td(x,y) and\ d(y,z) = (1-t)d(x,y).$$

We use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (i) above

(ii) For $x,y,z \in X$ *and* $t \in [0,1]$, we have

$$d\big((1-t)x \oplus ty, z\big) \le (1-t)d(x,z) + td(y,z).$$

(iii) For $x,y,z \in X$ *and* $t \in [0,1]$, we have

$$d^2\big((1-t)x \oplus ty, z\big) \le (1-t)d^2(x,z) + td^2(y,z) - t(1-t)d^2(x,y).$$

Lemma 1.1.8 Qihou,L. 2001 .Let $\{a_n\},\{\lambda_n\}$ *and* $\{c_n\}$ be the sequences of nonnegative numbers such that $a_n + \le (1+\lambda_n)\, a_n + c_n,$

For all $n \ge 1.\ If\ \sum_{n=1}^{\infty} \lambda_n < \infty\ and\ \sum_{n=1}^{\infty} c_n < \infty,\ then\ \lim_{n\to\infty} a_n$ exists. Whenever, if there exists a subsequence $\{a_{ni}\} \subseteq \{a_n\}$ such that $a_{ni} \to 0\ as\ i \to \infty, then\ \lim_{n\to\infty} a_n = 0.$

Lemma 1.1.9 Laokul and Panyanak 2010 . Let (X,d) be a complete CAT(0) space and $x \in X$. Suppose $\{t_n\}$ is a sequence in $[a,b\,]$ for some $a,b \in (0,1)$ *and* $\{u_n\},\{v_n\}$ are two sequences in X such that

$$\lim_{n\to\infty} sup\, d(u_n,x) \le c$$

$$\lim_{n\to\infty} \sup d(v_n,x) \le c$$

$$\lim_{n\to\infty} sup\, d((1-t_n)u_n \oplus t_n v_n, x) = c$$

For some $c \ge 0$.Then $\lim_{n\to\infty} \sup d(u_n,v_n) = 0.$

Definition 2 .Berinde 2007. let $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ be two real sequences of real numbers with limit a and b respectively. Assume that there exists $\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = l$

(i) $if\ l = 0$ then we say that $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$ and

(ii) $if\ 0 < l < \infty$ then we say that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of converges.

Convergences Analysis

Theorem 4.1 let C be a nonempty closed, bounded and convex subset of a complete $CAT(0)$ space $X\ and\ let\ T : C \to C$ be a uniformly $L - Lipschitzian$ and $((\{\mu_n\}, \{k_n\}\ \varphi\ )-$ total asymptotically nonexpansive mapping. Suppose that the following conditions are satisfied

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} k_n < \infty$

(ii) $there\ exist\ constants\ b, d\ with\ 0 < b \leq \alpha_n \leq d < 1\ for\ each\ n \in N;$

(iii) $there\ exist\ constants\ a, c\ with\ 0 < a \leq \beta_n \leq c < 1\ for\ each\ n \in N;$

(iv) $there\ exist\ a\ constant\ M^* such\ that\ \psi(r) \leq M^*\ for\ r \geq 0.$

Then the sequence $\{u_n\}$ defined by (1.2) $\Delta - converges$ to a fixed point of $F(T)$.

Proof. We first note that $F(T) \neq 0$ by lemma 1.1.1. We divide the proof of Theorem 4.1 into three steps.

Step-I. first we prove that $\lim_{n\to\infty} d(u_n, p)$ exists for any $p \in F(T)$.

Step-II. We prove that

$$\lim_{n\to\infty} d\,(u_n, Tu_n) = 0.$$

Step-III. We show that the sequence $\{u_n\}$ $\Delta -$ converges to a fixed point of T. We prove that,

$$w_\Delta\{u_n\} = \cup_{\{v_n\}\subseteq\{u_n\}} A(\{v_n\}) \subseteq F(T)$$

And $w_\Delta\{u_n\}$ consists of exactly one point

Proof of step-I: where $\{u_n\}$ is defined by (1.2). Let $p \in F(T)$. Then we have

$$d(w_n, p) = d((1-\alpha_n)u_n \oplus \alpha_n T^n u_n,, p)$$

$$\leq (1-\alpha_n)d(u_n, p) + \alpha_n d(T^n u_n, p)$$

$$\leq (1-\alpha_n)d(u_n, p) + \alpha_n(d(u_n, p) + \mu_n\varphi\big(d(u_n, p)\big) + k_n)$$

$$= d(u_n, p) + \alpha_n\big[\mu_n\varphi\big(d(u_n, p)\big) + k_n\big]$$

$$\leq d(u_n, p) + \mu_n \varphi\big(d(u_n, p)\big) + k_n$$

$$\leq (1 + \mu_n M^*) d(u_n, p) + k_n \qquad 4.1$$

For all $n \in N$. Also, we have

$$d(v_n, p) = d(T^n\big((1 - \beta_n) w_n \oplus \beta_n T^n w_n, p\big))$$

$$\leq d\big((1 - \beta_n) w_n \oplus \beta_n T^n w_n, p\big) + \mu_n \varphi\left(d\big((1 - \beta_n) w_n \oplus \beta_n T^n w_n, p\big)\right) + k_n$$

$$\leq \Big((1 + \mu_n M^*) d\big((1 - \beta_n) w_n \oplus \beta_n T^n w_n\big), p\Big) + k_n$$

$$\leq (1 + \mu_n M^*)\big((1 - \beta_n) d(w_n, p) + \beta_n d(T^n w_n, p)\big) + k_n$$

$$\leq (1 + \mu_n M^*)\big((1 - \beta_n) d(w_n, p) + \beta_n (d(w_n, p) + \mu_n \varphi d(w_n, p) + k_n)\big) + k_n$$

$$\leq (1 + \mu_n M^*)\big((1 + \mu_n M^*) d(w_n, p) + k_n\big) + k_n$$

$$\leq (1 + \mu_n M^*)^2 d(w_n, p) + (2 + \mu_n M^*) k_n$$

$$\leq (1 + \mu_n M^*)^2 [(1 + \mu_n M^*)^2 d(u_n, p) + (2 + \mu_n M^*) k_n] + (2 + \mu_n M^*) k_n$$

$$\leq (1 + \mu_n M^*)^4 d(u_n, p) + (1 + \mu_n M^*)^2 (2 + \mu_n M^*) k_n + (2 + \mu_n M^*) k_n$$

$$\leq (1 + \mu_n M^*)^4\, d(u_n, p) + (2 + \mu_n M^*)(1 + (1 + \mu_n M^*)^2) k_n \qquad 4.2$$

For each $n \in N$. From (1.2), (4.1) and (4.2), we get

$$d(u_{n+1}, p) = d(T^n v_n, p)$$

$$\leq d(v_n, p) + \mu_n \varphi d(v_n, p) + k_n$$

$$\leq (1 + \mu_n M^*) d(v_n, p) + k_n$$

$$\leq (1 + \mu_n M^*)[(1 + \mu_n M^*)^4\, d(u_n, p) + (2 + \mu_n M^*)(1 + (1 + \mu_n M^*)^2) k_n] + k_n$$

$$\leq (1 + \mu_n M^*)^5\, d(u_n, p) + (1 + \mu_n M^*)(2 + \mu_n M^*)(1 + (1 + \mu_n M^*)^2) k_n + k_n$$

$$\leq (1 + \mu_n M^*)^5\, d(u_n, p) + [1 + (1 + \mu_n M^*)(2 + \mu_n M^*)(1 + (1 + \mu_n M^*)^2)] k_n$$

4.3

Where

$$\xi_n := (1 + \mu_n M^*)^5 \quad \text{and } \delta_n := 1 + (1 + \mu_n M^*)(2 + \mu_n M^*)(1 + (1 + \mu_n M^*)^2)$$

By assumption (i) , we have

$$\sum_{n=1}^{\infty} \xi_n < \infty \quad and \quad \sum_{n=1}^{\infty} \delta_n < \infty \qquad 4.4$$

By assertion (4.3) ,(4.4) and lemma 1.1.8 we obtain $\lim_{n\to\infty} d(u_n, p)$ exists.

Proof of steep- II. It follows from Step-I that $\lim_{n\to\infty} d(u_n, p)$ exists for each given $p \in F$. Without loss of generality, we can assume that

$$\lim_{n\to\infty} d(u_n, p) = r \geq 0 \qquad 4.5$$

From (2.1), we have

$$\lim_{n\to\infty} \sup d\,(w_n, p) \leq r \qquad 4.6$$

Since T is $(\mu_n, k_n, \psi)-$ total asymptotically nonexpansive mapping,

$$d(T^n w_n, p) \leq d(w_n, p) + \mu_n \varphi\big(d(w_n, p)\big) + k_n$$

$$\leq (1 + \mu_n M^*) d(w_n, p) + k_n \qquad 4.7$$

From (4.6) and (4.7),we have

$$\lim_{n\to\infty} \sup d\,(T^n w_n, p) \leq r \qquad 4.8$$

In the same way, we get

$$\lim_{n\to\infty} \sup d\,(T^n u_n, p) \leq r \qquad 4.9$$

Since

$$d(u_{n+1}, p) \leq (1 + \mu_n M^*)^5\, d(u_n, p) + [1 + (1 + \mu_n M^*)(2 + \mu_n M^*)(1 + (1 + \mu_n M^*)^2)] k_n$$

By taking limit infimum both sides, we obtain;

$$r \leq \lim_{n\to\infty} \inf d(w_n, p) \qquad 4.10$$

From (4.6) and (4.10) we obtain

$$r = \lim_{n\to\infty} sup\, d(w_n, p) = \lim_{n\to\infty} sup\, d\big((1 - \alpha_n) u_n \oplus \alpha_n T^n u_n, p\big) \qquad 4.11$$

$$d\big((1 - \alpha_n) u_n \oplus \alpha_n T^n u_n\big), p \leq [1 + \mu_n M^*] d\big((1 - \alpha_n) u_n \oplus \alpha_n T^n u_n, p\big) + k_n$$

$$\lim_{n\to\infty} sup\; d\big((1 - \alpha_n) u_n \oplus \alpha_n T^n u_n\big), p \leq \lim_{n\to\infty} sup\, d((1 - \alpha_n) w_n \oplus \alpha_n T^n u_n, p)$$

$$r \leq \lim_{n\to\infty} sup\, d((1 - \alpha_n) u_n \oplus \alpha_n T^n u_n, p) \qquad 4.12$$

By using (4.5) and (4.9), we have

$$d\big((1-\alpha_n)u_n \oplus \alpha_n T^n u_n, p\big) \le (1-\alpha_n)d(u_n,p) + \alpha_n d(T^n u_n, p)$$

$$\lim_{n\to\infty} sup\, d\big((1-\alpha_n)u_n \oplus \alpha_n T^n u_n, p\big) \le r \qquad 4.13$$

Applying (4.12) and (4.13), we have,

$$\lim_{n\to\infty} sup\, d\big((1-\alpha_n)u_n \oplus \alpha_n T^n u_n, p\big) = r \qquad 4.14$$

By combining (4.5), (4.9) (4.14) and using Lemma 1.1.9 we can conclude that

$$\lim_{n\to\infty} d\,(u_n, T^n u_n) = 0 \qquad 4.15$$

We also have

$$d(u_{n+1}, p) \le (1+\mu_n M^{\wedge}) d(v_n, p) + k_n$$

By taking limit infimum both sides we obtain

$$r \le \lim_{n\to\infty} inf\, d(v_n, p) \qquad 4.16$$

By (4.2), we have

$$d(v_n, p) \le (1+\mu_n M^*)^4\, d(u_n, p) + (2+\mu_n M^*)(1+(1+\mu_n M^*)^2)k_n$$

By taking limit suprimum both sides, we obtain

$$\operatorname{im}_{n\to\infty} sup\, d(v_n, p) \le r \qquad 4.17$$

By using (4.16) and (4..17), we get

$$r = \lim_{n\to\infty} sup\, d(v_n, p) = \lim_{n\to\infty} sup\, \big(T^n\big((1-\beta_n)w_n \oplus \beta_n T^n w_n\big), p\big) \qquad 4.18$$

$$\begin{aligned} d\big(T^n\big((1-\beta_n)w_n \oplus \beta_n T^n w_n\big), p\big) &\le d\big((1-\beta_n)w_n \oplus \beta_n T^n w_n, p\big) + \mu_n \varphi\big[d\big((1-\beta_n)w_n \oplus \beta_n T^n w_n, p\big)\big] \\ &\quad + k_n \\ &\le [1+\mu_n M^*] d\big((1-\beta_n)w_n \oplus \beta_n T^n w_n, p\big) + k_n, \end{aligned}$$

$$\lim_{n\to\infty} sup\, d\big(T^n\big((1-\beta_n)w_n \oplus \beta_n T^n w_n\big), p\big) \le \lim_{n\to\infty} sup\quad d\big((1-\beta_n)w_n \oplus \beta_n T^n w_n, p\big)$$

$$r \le \lim_{n\to\infty} inf\, d\big((1-\beta_n)w_n \oplus \beta_n T^n w_n, p\big) \qquad 4.19$$

Also we have

$$d\big((1-\beta_n)w_n \oplus \beta_n T^n w_n, p\big) \leq (1-\beta_n)d(w_n, p) + \beta_n d(T^n w_n, p)$$

$$\limsup_{n\to\infty} d\big((1-\beta_n)w_n \oplus \beta_n T^n w_n, p\big) \leq r. \qquad 4.20$$

By using (4.8) (4.11), (4.20) and lemma 1.1.9 we conclude that

$$\lim_{n\to\infty} d\,(w_n, T^n w_n) = 0 \qquad 4.21$$

Since T is a $(\mu_n, k_n, \psi)-$ total asymptotically nonexpansive mapping,

$$d(T^n w_n, T^n u_n) \leq d(w_n, u_n) + \mu_n \varphi d(w_n, u_n) + k_n$$

$$\leq (1 + \mu_n M^*) d(w_n, u_n) + k_n$$

$$= (1 + \mu_n M^*)\, d\big((1-\alpha_n)u_n \oplus \alpha_n T^n u_n,\ u_n\big) + k_n$$

$$= (1 + \mu_n M^*)[\alpha_n d(T^n u_n, u_n)] + k_n, \qquad \forall\, n \in N$$

By taking limit $n \to \infty$ using (4.15) we get

$$\lim_{n\to\infty} d\,(T^n w_n, T^n u_n) = 0 \qquad 4.22$$

We have

$$d(T^n v_n, T^n w_n) \leq d(v_n, w_n) + \mu_n \varphi d(v_n, w_n) + k_n$$

$$\leq (1 + \mu_n M^*) d(v_n, w_n) + k_n$$

$$\leq (1 + \mu_n M^*) d\big(T^n\big((1-\beta_n)w_n \oplus \beta_n T^n w_n\big), w_n\big) + k_n$$

$$\leq (1 + \mu_n M^*) d\big(T^n\big((1-\beta_n)w_n \oplus \beta_n T^n w_n\big), T^n w_n\big) + (1 + \mu_n M^*) d(T^n w_n, w_n) + k_n$$

$$\leq (1 + \mu_n M^*)[d\big(\big((1-\beta_n)w_n \oplus \beta_n T^n w_n\big), w_n\big) + \mu_n M^*\, d\big((1-\beta_n)w_n \oplus \beta_n T^n w_n\big), w_n + k_n] \quad + (1 + \mu_n M^*) d(T^n w_n, w_n) + k_n$$

$$\leq (1 + \mu_n M^*)^2 [\beta_n d(T^n w_n, w_n)] + (1 + \mu_n M^*) d(T^n w_n, w_n) + (2 + \mu_n M^*) k_n \quad \forall\ n \in N$$

By taking limit as $n \to \infty$ and using (4.21) we obtain

$$\lim_{n\to\infty} d\,(T^n v_n, T^n w_n) = 0 \qquad 4.23$$

From (4.15), (4.22) and (4.23), we get

$$d(u_n, u_{n+1}) = d(u_n, T^n v_n),$$

$$\leq d(u_n, T^n u_n) + d(T^n u_n, T^n w_n) + d(T^n w_n, T^n v_n)$$

$\to 0 \; as \; n \to \infty$

Since T is $(\mu_n, k_n, \psi) -$ total asymptotically nonexpansive mapping and uniformly $L - Lipschitzian,$ we obtain

$$d(u_n, Tu_{n+1}) = \quad d(u_n, u_{n+1}) + d(u_{n+1}, T^{n+1} u_{n+1}) + d(T^{n+1} u_{n+1}, T^{n+1} x_n) + d(T^{n+1} x_n, T x_n),$$

$$\leq \quad d(u_n, u_{n+1}) + d(u_{n+1}, T^{n+1} u_{n+1}) + L d(u_{n+1}, x_n) + L d(T^n x_n, x_n),$$

$$\to 0 \; as \; n \; \to \; \infty.$$

Proof of Step-III. Next, we claim that the sequence $\{u_n\}$ $\Delta -$converges to a fixed point of $T.$ Indeed, we will show that

$$w_\Delta\{u_n\} \; = \bigcup_{\{v_n\} \subseteq \{u_n\}} A(\{v_n\}) \subseteq F(T)$$

$w_\Delta\{u_n\}$ Consists of exactly one point. Let $v \in w_\Delta\{u_n\}$. By the definition of $w_\Delta\{u_n\}$, there exists subsequence $\{v_n\} \; of \; \{u_n\}$ such that $A(\{v_n\}) = \{v\},$ from Lemma 1.1.2,there is a subsequence $\{u_n\} \; of \; \{v_n\}$ such that $\Delta - \lim_{n\to\infty} u_n = u \; and \; u \in C.$ By Lemma 1.1.5,we have $u \in F(T).$ Since $\{d(v_n, u)\}$ converges by Lemma 1.1.4 we get $v = u.$ Thus $w_\Delta\{u_n\} \subseteq F(T).$

Finally, we prove that $w_\Delta\{u_n\}$ consists of exactly one point. Let $\{v_n\}$ be a subsequence of $\{u_n\}$ by the uniqueness asymptotic center such that $A(\{v_n\}) = v$ and let $A(\{u_n\}) = \{u\}$, Since $v = u \; \in F(T) \; and \; \{d(v_n, u)\}$ converges ,by using Lemma 1.1.4 we see that $u \; = v \in F(T).$ Therefore $w_\Delta\{u_n\} = \{u\}.$ This completes the proof.

We now establish some strong convergence results

Theorem 4.2 let $X, C, T, \{\alpha_n\}, \{\beta_n\}, \{u_n\}$ *satisfy the hypothesis of* ***Theorem 4.1****. Then the sequence* $\{u_n\}$ *generated by (1.2) converges strongly to a fixed point of T if and only if* $\lim_{n\to\infty} inf \;\; d(u_n, F(T)) = 0.$

$$where \; d(u, F(T)) = inf\{d(u, p) : p \in F(T)\},$$

Proof : Conversely, suppose that $\lim_{n\to\infty} inf\ d(u_n, F(T)) = 0$. As proved in step-1 of theorem 3.1, $\lim_{n\to\infty} d\left(u_n, F(T)\right)$ *exists for all* $p \in F(T)$. Thus, by hypothesis, $\lim_{n\to\infty} d\ (u_n, F(T)) = 0$.

Next, we show that $\{u_n\}$ is a Cauchy sequence in C. Let $\varepsilon \geq 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d\left(u_n, F(T)\right) = 0$, there exists a positive integer n_0 such that for all $n \geq n_0$,

$$d(U_N, F(T)) < \frac{\varepsilon}{4}$$

In particular, $inf\{d(x_n, p): p \in F(T)\} < \frac{\varepsilon}{4}$,Thus, there exists $p^* \in F(T)$ such that

$$d(u_{n_0}, p^*) < \frac{\varepsilon}{2},$$

Now, for all $m, n \geq n_0$, we have

$$d(u_{n+m}, u_n) \leq d(u_n, p^*) + 2d\left(u_{n_0}, p^*\right) < 2\left(\frac{\varepsilon}{2}\right) = \varepsilon,$$

i.e., $\{u_n\}$ is a Cauchy sequence in the closed interval C of a complete CAT (0) space and hence it converges to appoint q *in* C. Now, $\lim_{n\to\infty} d\left(u_n, F(T)\right) = 0$ gives that $d(q, F(T)) = 0$ and closedness of *F(T)* forces q *to be in* $F(T)$. This completes the proof.

The concept of special self mapping is called Condition (I) proposed by Senter and Dotson 1974. as follows

Definition 3. let (X, d) be a CAT(0) space and K a nonempty subset. A self mapping T with $F(T) \neq 0$ is said to satisfy condition (I) if there is a non-decreasing function $f: [0, \infty) \to [0, \infty)$ with $f(0) = 0$ *and* $f(l) > 0$ *for all* $l > 0$ *such that* $d(x, Tx) \geq f\left(d(x, F(T))\right)$ *forall* $x \in K$.

Using condition(I), we now establish the following strong convergence result for total asymptotically nonexpansive mapping.

Theorem 4.3 **Let** $X, C, T, \{\alpha_n\}, \{\beta_n\}, \{u_n\}$ satisfy the hypothesis of ***Theorem 4.1 and*** let T be a mapping satisfying condition (I). Then the sequence $\{u_n\}$ generated by (1.2) converges strongly to a fixed point of *T*.

Proof . as proved in theorem 4.2, $\lim_{n\to\infty} d\left(u_n, F(T)\right) exist.$ Also, by step –II of Theorem 4,1, we have $\lim_{n\to\infty} d\,(u_n, Tu_n) = 0.$ It follows from condition *(I)* that

$$\lim_{n\to\infty} f\left(d\left(u_n, F(T)\right)\right) \leq \lim_{n\to\infty} d\,(u_n, Tu_n) = 0.$$

That is $\lim_{n\to\infty} f\left(d\left(u_n, F(T)\right)\right) = 0, Since\ f: [0, \infty) \to [0, \infty)$ is a non-decreasing function satisfying $f(0) = 0\ and\ f(r) > 0\ for\ all\ r > 0,$ we obtain

$$\lim_{n\to\infty} d\,(u_n, F(T)) = 0$$

Now all the conditions of the above Theorem are satisfied, therefore by its conclusion $\{x_n\}$ converges strongly to a point of $F(T)$.

Discussion of results

Our theorem 3.1 extends theorem 5.7 of Nanjaras and Panyanak (2010) and theorem 3.5 of Niwongsa and Panyanak (2011) to the case of more general class of asymptotically nonexpansive mappings. It also extends theorem 3.1 of Manoj and Hemant (2022) it contains theorem 2.1 of Ahmed and Sabah (2022) and theorem 3.1 of Anantachai and Pakeeta (2023).

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