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A Simplified Hybrid Analytical Method for Solving Integer and Fractional-Order Differential Equations without Adomian Polynomials or Lagrange Multipliers

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Abstract

In this study, we propose a novel hybrid analytical technique that combines the Adomian Decomposition Method (ADM) with the Variational Iteration Method (VIM) to solve a class of linear and nonlinear first-order initial value problems (IVPs), including those of fractional order. The principal aim of this approach is to overcome the computational challenges typically encountered in each individual method—namely, the complexity of generating Adomian polynomials in ADM and the requirement for Lagrange multipliers in VIM. By synthesizing the strengths of both methods, the hybrid scheme constructs analytical series solutions without necessitating linearization, Adomian polynomials, or the explicit formulation of Lagrange multipliers. This significantly streamlines the solution process while preserving accuracy and generality. The validity and computational efficiency of the proposed method are substantiated through a series of illustrative examples, encompassing both integer-order and fractional differential equations. The results demonstrate that the hybrid approach not only simplifies implementation but also yields precise and rapidly converging solutions, making it a robust alternative for tackling a broad spectrum of initial value problems in mathematical modeling and applied sciences.

Keywords: Fractional differential equations; Initial value problems; Adomian Decomposition Method; Variational Iteration Method; Hybrid analytical methods; Series solutions; Nonlinear differential equations; Computational simplification.

Introduction

Fractional and ordinary differential equations play a fundamental role in modeling various phenomena in engineering and applied sciences. Over the past few decades, considerable effort has been devoted to developing analytical and approximate methods for solving both linear and nonlinear differential equations. Notable among these are the Adomian Decomposition Method (ADM) (Tate & Dinde, 2019), the Variational Iteration Method (VIM) (Wazwaz, 2009), the Homotopy Perturbation Method (HPM) (Momani & Odibat, 2007), and the Differential Transform Method (DTM) (Rashidi et al., 2020; Bhalekar & Daftardar-Gejji, 2012).

While these methods have yielded valuable insights and approximate solutions, they are not without limitations. Specifically, ADM requires the often cumbersome computation of Adomian polynomials for nonlinear terms (Hemeda, 2018), and VIM can suffer from slow convergence due to the integration-by-parts process involved in determining the Lagrange multiplier (Ziane & Cherif, 2018).

This work introduces a modified hybrid method that circumvents these challenges. Our approach avoids both linearization and the use of Adomian polynomials and Lagrange multipliers. We extend the method to fractional-order differential equations and analyze how the solutions behave as the fractional order approaches an integer.

Hybridized Scheme for the Solutions of Nonlinear Ordinary Differential Equations

In this section we proposed a new scheme for solving ordinary differential equations. Our method is based on the classical ADM combine with the VIM.

Consider the following nonlinear ordinary differential equations:

$$L[u(t)] = N[u(t)] + g(t), \quad \ldots (1)$$

Where L and N are linear and nonlinear operators, respectively, $g(t)$ is the source inhomogeneous term.

To illustrate the hybrid concepts, we first construct the correction functional as follows using the Adomian recursive relations:

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1}\sum_{k=0}^{\infty}(L[u(t)] - N[u(t)] - g(t))\right] dt \quad \ldots (2)$$

Where

$$\sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} \quad \ldots (3)$$

Is obtained from the given initial condition(s)

From this equation, the iterate are determined by the following recursive way

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} \quad \ldots (4)$$

$$u_{n+1}(x) = (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1}\sum_{k=0}^{n}\left(Lu_k(t) + Nu_k(t)\right) - g(t)\right] dt \quad \ldots (5)$$

$n \geq 0$

Where q is determined by the order of the equation under consideration

The Hybrid method for the Solutions of Nonlinear Fractional First-order Differential Equations.

Consider the following nonlinear fractional differential equations:

$$D^{\alpha}[u(t)] + L[u(t)] + N[u(t)] = g(t), \quad t > 0 \qquad \ldots (6)$$

$where\ L$ is a linear operator, N represent a nonlinear operator, $g(t)$ is the source term, and D^{α} is the Caputo fractional derivative operator of order α with $m-1 < \alpha < m.$

We extend reliable modifications of the Adomian decomposition method (ADM) presented in (Odibat, 2019) by using the ADM to reconstruct the VIM for the solutions of nonlinear fractional differential equations.

The correctional functional for Eqn. (6) is approximately expressed as follows by applying the integral operator and the *n-fold* integral formula to both sides of Eq. (6), we obtained the solution as follows:

$$u(x) = \sum_{k=0}^{m-1} u^{k}(0)\frac{x^{k}}{k!} + (-1)^{q}\frac{1}{(q-1)!}\int_{0}^{x}[(t-x)^{q-1}(D^{\alpha}[u(t)] + L[u(t)] + N[u(t)] - g(t))]dt \qquad \ldots (7)$$

Now, we rewrite the correctional function Eqn. (7) in Adomian recursive relations

$$u_{n+1}(x) = \sum_{k=0}^{m-1} u^{k}(0)\frac{x^{k}}{k!} + (-1)^{q}\frac{1}{(q-1)!}\int_{0}^{x}\left[(t-x)^{q-1}\sum_{k=0}^{\infty}\left(D^{\alpha}[u_{k}(t)] + Lu_{k}(t) + Nu_{k}(t)\right) - g(t)\right]dt \qquad \ldots (8)$$

$n \geq 0.$

Now, we set the following scheme:

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} \quad \ldots (9)$$

$$u_1(x) = (-1)^q \frac{1}{(q-1)!}\int_0^x [(t-x)^{q-1}(D^\alpha[u_0(t)] + Lu_0(t) + Nu_0(t) - g(t)]dt \quad \ldots (10)$$

$$u_2(x) = (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1}\sum_{k=0}^{1}\left(D^\alpha[u_k(t)] + Lu_k(t) + Nu_k(t)\right) - g(t)\right]dt \quad \ldots (11)$$

$$u_3(x) = (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1}\sum_{k=0}^{2}\left(D^\alpha[u_k(t)] + Lu_k(t) + Nu_k(t)\right) - g(t)\right]dt \quad \ldots (12)$$

$$u_n(x) = (-1)^q \frac{1}{(q-1)!}\int_0^x [(t-x)^{q-1}\sum_{k=0}^{n-1}\left(D^\alpha[u_k(t)] + Lu_k(t) + Nu_k(t)\right) - g(t)]dt \quad \ldots (13)$$

And the series solution is given as

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + u_1(x) + u_2(x) + u_3(x) + u_4(x) + \cdots \quad \ldots (14)$$

Implementation of the Hybrid Scheme

This section is performed to explore the performance and the accuracy of the suggested hybrid method as derived in **3.0**, for solving nonlinear first order ODEs of integer and non-integer order. Test problems have been solved by means of ODM, ADM, VIM and the suggested hybrid method. Then numerical comparisons between the hybrid method solutions, NIM, VIM, ODM and ADM solutions are made.

The differential Logistic Equation.

Example 1:

$$D^{\alpha}u(x) = \frac{1}{2}u(x)\big[\big(1-u(x)\big)\big],\;\; x \geq 0,$$

$$0 < \alpha \leq 1\,, \qquad \ldots (15)$$

Subject to the initial condition

$$u(0) = \frac{1}{2}. \qquad \ldots (16)$$

With exact solution $u(x) = \frac{e^{\frac{1}{2}x}}{1+e^{\frac{1}{2}x}}$

In view of Eqn. (7), the correctional functional for Eqn. (15) is approximately expressed as follows:

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + (-1)^q\frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1}\left(D^{\alpha}u(t) - \frac{1}{2}u(t)[1-u(t)]\right)\right]dt \qquad \ldots (17)$$

Now, we rewrite the correctional function Eqn. (17) in Adomian recursive relations as follows:

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} = \frac{1}{2} \qquad \ldots (18)$$

$$u_{n+1}(x) = (-1)^q\frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1}\sum_{k=0}^{\infty}\left(D^{\alpha}u_k(t) - \frac{1}{2}u_k(t)[1-u_k(t)]\right)\right]dt \qquad \ldots (19)$$

$n \geq 0.$

$$u_1(x) = (-1)^q \frac{1}{(q-1)!} \int_0^x \left[(t-x)^{q-1} \left(D^\alpha u_0(t) - \frac{1}{2} u_0(t)[1-u_0(t)] \right) \right] dt$$

$$= \frac{1}{8} \frac{x^\alpha}{\Gamma(\alpha+1)} \qquad \dots (20)$$

$$u_2(x) = (-1)^q \frac{1}{(q-1)!} \int_0^x \left[(t-x)^{q-1} \sum_{k=0}^{1} \left(D^\alpha u_k(t) - \frac{1}{2} u_k(t)[1-u_k(t)] \right) \right] dt =$$

$$-\frac{1}{8} \frac{x^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{1}{8} x - \frac{1}{384} x^3$$

$\dots (21)$

$$u_3(x) = (-1)^q \frac{1}{(q-1)!} \int_0^x \left[(t-x)^{q-1} \sum_{k=0}^{2} \left(D^\alpha [u_k(t)] + u_k^2(t) - 1 \right) \right] dt =$$

$$-\frac{1}{8} \frac{x^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{1}{64} \frac{x^{4-\alpha}}{\Gamma(5-\alpha)} + \frac{1}{8} x - \frac{1}{2064384} x^7 + \frac{1}{15360} x^5 - \frac{1}{384} x^3$$

$+\dots \qquad \dots (22)$

So that the analytic solution is

$$u(x) = u_0(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x) \cong \frac{1}{2} + \frac{1}{4} x - \frac{1}{192} x^3 - \frac{1}{8} \frac{x^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{1}{64} \frac{x^{4-\alpha}}{\Gamma(5-\alpha)} - \frac{1}{2064384} x^7 + \frac{1}{15360} x^5$$

……(23)

Table 1: Approximate solution of Example 1with $\alpha=1$

X	EXACT	HYBRID METHOD (α=1, n=3)	VIM (α=1, n=6)	ADM (α=1, n=5)	NIM (α=1, n=5)
0	0.5	0.5	0.5	0.5	0.5
0.1	0.5125	0.512497396	0.524979187	0.512497396	0.512497396
0.2	0.525	0.524979187	0.549833997	0.524979187	0.524979187
0.3	0.5374	0.537429846	0.574442517	0.537429846	0.537429846
0.4	0.5498	0.549833999	0.59868766	0.549834	0.549833999
0.5	0.5622	0.56217651	0.622459332	0.562176514	0.56217651
0.6	0.5744	0.574442549	0.645656314	0.574442563	0.574442549

X	EXACT	HYBRID METHOD (α=1, n=3)	VIM (α=1, n=6)	ADM (α=1, n=5)	NIM (α=1, n=5)
0.7	0.5866	0.586617673	0.668187813	0.586617713	0.586617673
0.8	0.5987	0.598687898	0.689974656	0.598688	0.598687898
0.9	0.6106	0.610639774	0.71095013	0.610640006	0.610639774
1	0.6225	0.622460453	0.731060543	0.622460938	0.622460453

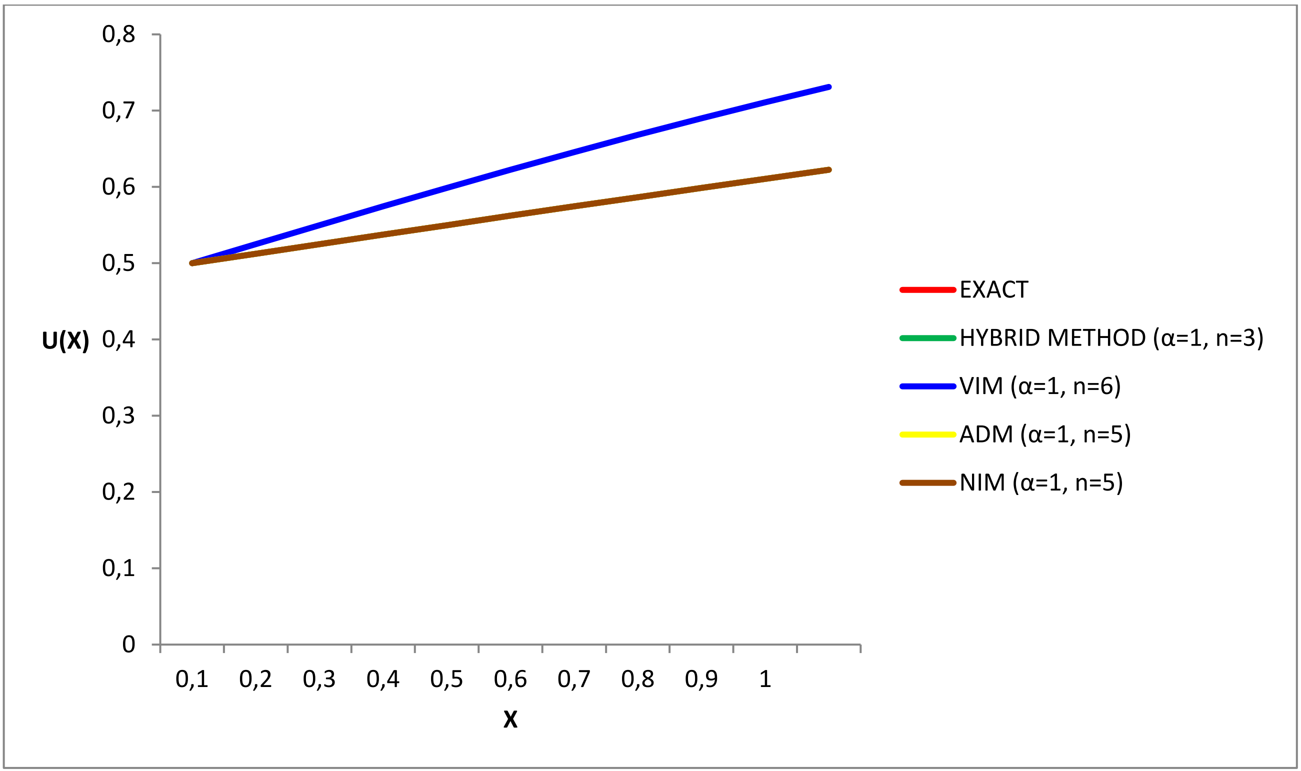

Figure 1: Comparison of the approximate solutions obtained by the Hybrid method, VIM, ADM and NIM with exact solution for example 1

Table 2: Numerical values when α=0.9, 0.8, 0.7 and 1.0 for Eq. (15)

X	EXACT (α=1)	HYBRID (α=1)	HYBRID (α=0.9)	HYBRID (α=0.8)	HYBRID (α=0.7)
0	0.5	0.5	0.5	0.5	0.5
0.1	0.512497396	0.512497396	0.515508571	0.517837823	0.519626022
0.2	0.524979187	0.524979187	0.529635909	0.533524717	0.536745423
0.3	0.537429845	0.537429846	0.54314496	0.548150559	0.552495001
0.4	0.549833997	0.549833999	0.556205668	0.561993248	0.567197618

X	EXACT (α=1)	HYBRID (α=1)	HYBRID (α=0.9)	HYBRID (α=0.8)	HYBRID (α=0.7)
0.5	0.562176501	0.56217651	0.568894306	0.575188011	0.581018305
0.6	0.574442517	0.574442549	0.581251422	0.587813578	0.594057342
0.7	0.586617579	0.586617673	0.593300105	0.599920238	0.606381597
0.8	0.59868766	0.598687898	0.605053731	0.611542011	0.618038433
0.9	0.610639234	0.610639774	0.61651986	0.622702857	0.629062929
1	0.622459331	0.622460453	0.627702436	0.633420216	0.639482055

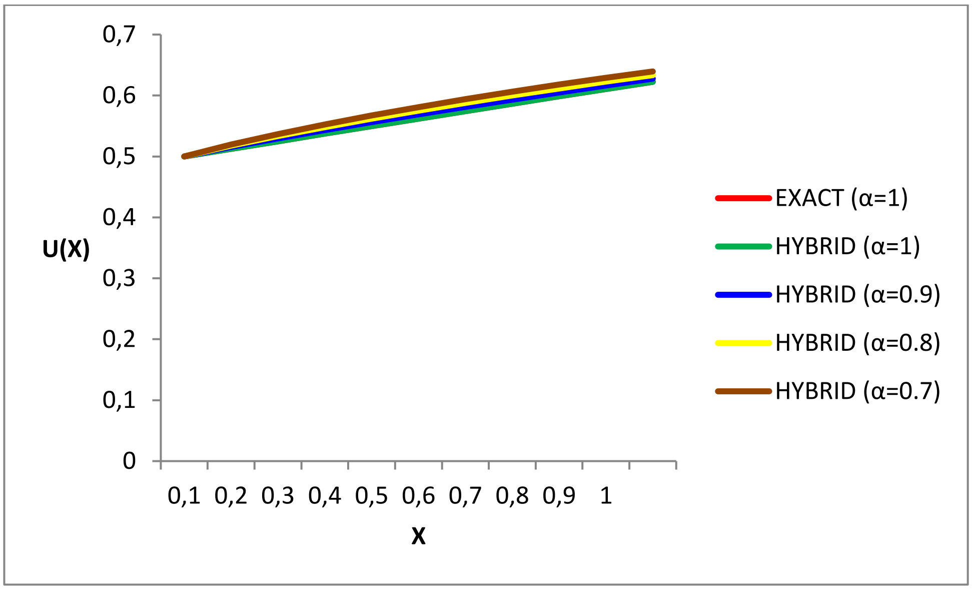

Figure 2: Comparison of the approximate solutions obtained by the Hybrid method for different values of α with the exact solution when α=1 for example 1

Nonlinear First Order Fractional Differential Equation.

Example 2:

$$D^{\alpha}u(x) + u^2(x) = 1, \qquad x \geq 0,$$

$$0 < \alpha \leq 1 \qquad \ldots(24)$$

Subject to the initial condition

$$u(0) = 0. \qquad \ldots(25)$$

With exact solution $u(x) = \tanh(x)$

In view of Eqn. (7), the correctional functional for Eqn. (24) is approximately expressed as follows:

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + (-1)^q \frac{1}{(q-1)!}\int_0^x [(t-x)^{q-1}(D^\alpha u(t) + u^2(t) - 1)]dt \quad \dots (26)$$

Now, we rewrite the correctional function Eqn. (26) in Adomian recursive relations as follows:

$$u_{n+1}(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1}\sum_{k=0}^{\infty}(D^\alpha u_k(t) + u^2{}_k - 1)\right]dt \quad \dots (27)$$

$n \geq 0.$

Now, we set the following scheme:

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} = 0 \quad \dots (28)$$

$$u_1(x) = (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1}(D^\alpha[u_0(t)] + u_0^2(t) - 1\right]dt = x \quad \dots (29)$$

$$u_2(x) = (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1}\sum_{k=0}^{1}\left(D^\alpha[u_k(t)] + u_k^2(t) - 1\right)\right]dt = x - \frac{x^3}{3} - \frac{x^{2-\alpha}}{\Gamma(3-\alpha)} \quad \dots (30)$$

$$u_3(x) = (-1)^q \frac{1}{(q-1)!} \int_0^x \left[(t-x)^{q-1} \sum_{k=0}^{2} \left(D^\alpha [u_k(t)] + u_k^2(t) - 1 \right) \right] dt$$
$$= x - \frac{4x^3}{3} + \frac{4x^5}{15} - \frac{x^7}{63} + \frac{4\Gamma(4-\alpha)x^{4-\alpha}}{\Gamma(3-\alpha)\Gamma(5-\alpha)} - \frac{2\Gamma(6-\alpha)x^{6-\alpha}}{3\Gamma(3-\alpha)\Gamma(7-\alpha)}$$
$$- \frac{\Gamma(5-2\alpha)x^{5-2\alpha}}{\Gamma(3-\alpha)\Gamma(6-2\alpha)} - \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{2x^{4-\alpha}}{\Gamma(5-\alpha)}$$
$$+ \frac{x^{3-2\alpha}}{\Gamma(4-2\alpha)} \qquad \ldots (31)$$

So that the analytic solution is

$$u(x) = u_0(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x)$$
$$\cong 3x - \frac{5x^3}{3} + \frac{4x^5}{15} - \frac{x^7}{63} + \frac{x^{3-2\alpha}}{\Gamma(4-2\alpha)} - \frac{3x^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{4\Gamma(4-\alpha)x^{4-\alpha}}{\Gamma(3-\alpha)\Gamma(5-\alpha)}$$
$$- \frac{2\Gamma(6-\alpha)x^{6-\alpha}}{3\Gamma(3-\alpha)\Gamma(7-\alpha)} - \frac{4x^3}{3} + \frac{4x^5}{15} - \frac{x^7}{63} + \frac{4\Gamma(4-\alpha)x^{4-\alpha}}{\Gamma(3-\alpha)\Gamma(5-\alpha)}$$
$$- \frac{2\Gamma(6-\alpha)x^{6-\alpha}}{3\Gamma(3-\alpha)\Gamma(7-\alpha)} + \frac{2x^{4-\alpha}}{\Gamma(5-\alpha)}$$
$$- \frac{\Gamma(5-2\alpha)x^{5-2\alpha}}{\Gamma(3-\alpha)\Gamma(6-2\alpha)} \qquad \ldots (32)$$

Table 3: Approximate solution of Example 2 with α=1

X	EXACT SOLN	HYBRID (α=1, n=3)	ODM (α=1, n=9)	ADM (α=1, n=12)
0	0	0	0	0
0.1	0.099668	0.099667995	0.0996680	0.099667997
0.2	0.197375	0.19737532	0.1973760	0.197375553
0.3	0.291313	0.291312612	0.2913240	0.291316363
0.4	0.379949	0.379948962	0.3800320	0.379974786
0.5	0.462117	0.462117157	0.4625000	0.462227183
0.6	0.53705	0.537049567	0.5383680	0.537390446
0.7	0.604368	0.604367777	0.6080760	0.605200136
0.8	0.664037	0.664036783	0.6730240	0.665700612
0.9	0.716298	0.716298153	0.7357320	0.719029569
1	0.761594	0.761598747	0.8000000	0.765079365

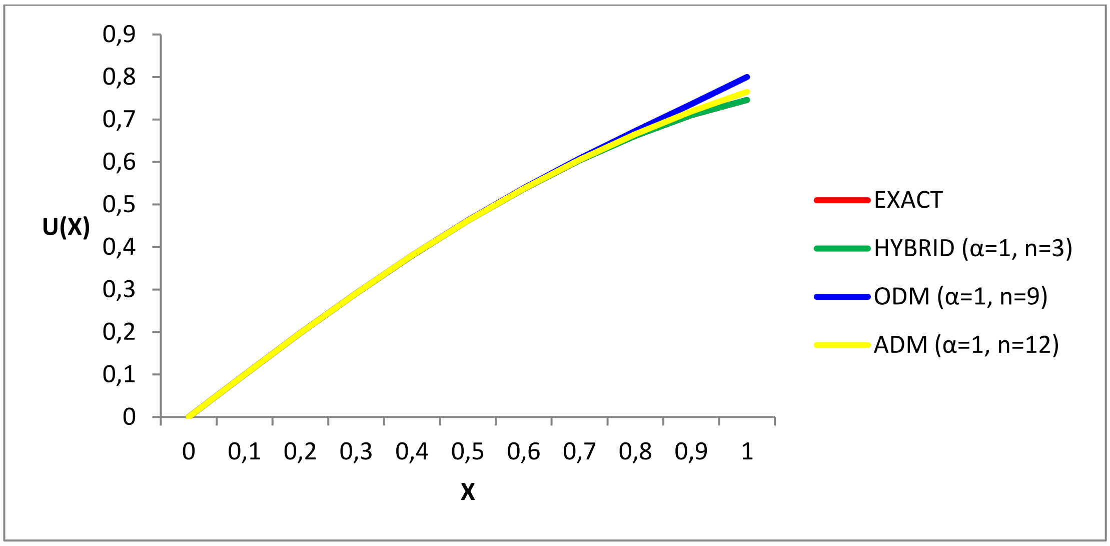

Figure 3: Comparison of the approximate solutions obtained by the Hybrid method, ODM, ADM and the exact solution for example 2

Table 4: Error Table of Example 2

ERROR OF APPRX SOLN	ERROR OF ODM	ERROR OF ADM
0	0	0
8.32667E-17	5.37504E-09	1.88298E-09
0	6.79775E-07	2.32791E-07
1.66533E-16	1.13875E-05	3.75041E-06
1.66533E-16	8.30377E-05	2.58238E-05
4.35207E-14	0.000382843	0.000110025
5.76239E-12	0.001318433	0.000340879
3.53712E-10	0.003708223	0.000832358
1.23864E-08	0.00898723	0.001663842
2.82443E-07	0.01943413	0.002731698
4.59102E-06	0.038405844	0.003485209

Table 5: Numerical values when α=0.9, 0.8, 0.7 and 1.0 for Eq. (15)

X	EXACT	HYBRID (α=1)	HYBRID (α=0.9)	HYBRID (α=0.8)	HYBRID (α=0.7)
0	0	0	0	0	0
0.1	0.099668	0.099667995	0.104824794	0.116634828	0.130573859
0.2	0.197375	0.19737532	0.201613799	0.215233465	0.234069
0.3	0.291313	0.291312612	0.291564035	0.302108771	0.320118738
0.4	0.379949	0.379948962	0.37353468	0.377189139	0.389728765
0.5	0.462117	0.462117157	0.446696128	0.440211885	0.443279055
0.6	0.53705	0.537049567	0.510816554	0.491410326	0.481538605
0.7	0.604368	0.604367777	0.56636346	0.531718609	0.505969675
0.8	0.664037	0.664036783	0.614530946	0.562824804	0.518811394
0.9	0.716298	0.716298153	0.657207765	0.587136433	0.523047622
1	0.761594	0.761598747	0.696884582	0.607672164	0.52228684

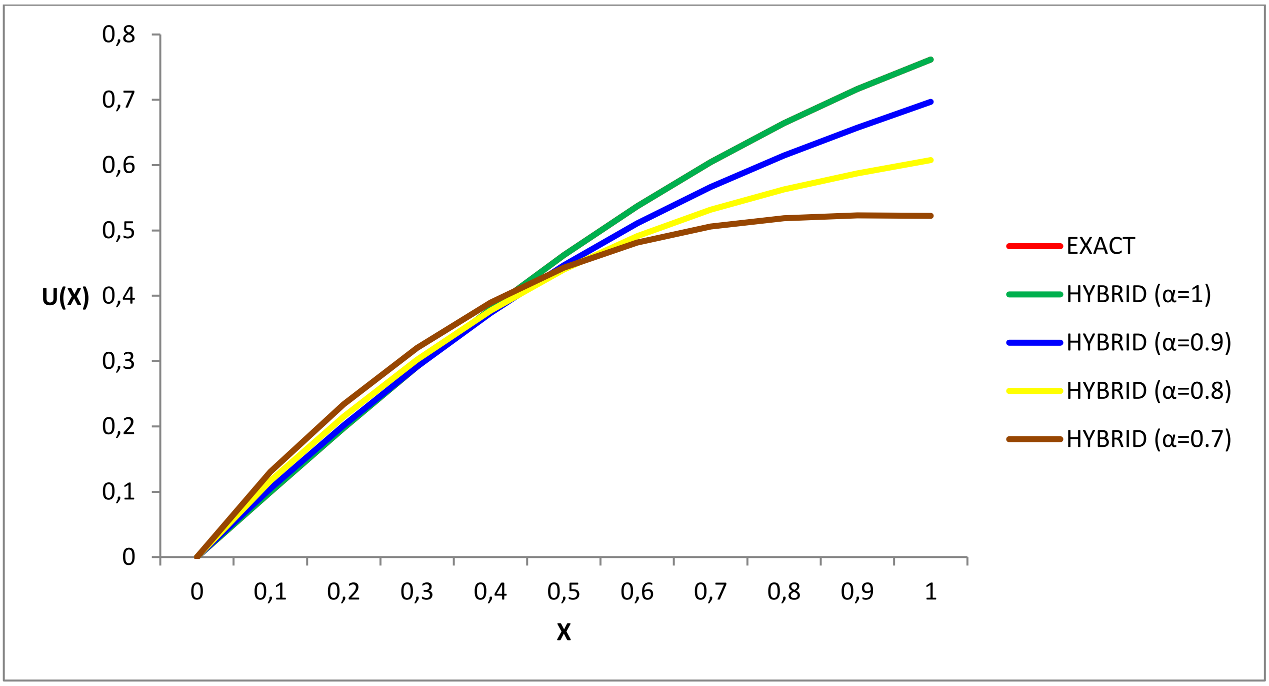

Figure 4: Comparison of the approximate solutions obtained by the Hybrid method with exact solution when α=1, 0.9, 0.8, and 0.7 of example 1

Linear First Order Fractional Differential Equation.

Example 3: Consider the first order fractional homogeneous ODE

$$D^{\alpha}u(x) - 2xu(x) = 0, \qquad x \geq 0, \qquad 0 < \alpha \leq 1 \qquad \dots (33)$$

Subject to the initial condition

$$u(0) = 1. \qquad \dots (34)$$

With exact solution $u(x) = e^{x^2}$ ... (35)

In view of Eqn. (7), the correctional functional for Eqn. (33) is approximately expressed as follows:

$$u(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + (-1)^q \frac{1}{(q-1)!}\int_0^x [(t-x)^{q-1}(D^{\alpha}u(t) - 2tu(t))]dt \qquad \dots (36)$$

Now, we rewrite the correctional function Eqn. (36) in Adomian recursive relations as follows:

$$u_n(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1}\sum_{k=0}^{\infty}\left(D^{\alpha}u_k(t) - 2tu_k(t)\right)\right]dt \qquad \dots (37)$$

$n \geq 0.$

Now, we set the following scheme:

$$u_0(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} = 1 \qquad \dots (38)$$

$$u_1(x) = (-1)^q \frac{1}{(q-1)!}\int_0^x [(t-x)^{q-1}(D^\alpha[u_0(t)] - 2tu_0(t)]dt$$

$$= x^2 \qquad \dots (39)$$

$$u_2(x) = (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1}\sum_{k=0}^{1}\left(D^\alpha[u_k(t)] - 2tu_k(t)\right)\right]dt$$

$$= x^2 + \frac{x^4}{2} - \frac{2x^{3-\alpha}}{\Gamma(4-\alpha)} \qquad \dots (40)$$

$$u_3(x) = (-1)^q \frac{1}{(q-1)!}\int_0^x \left[(t-x)^{q-1}\sum_{k=0}^{2}\left(D^\alpha[u_k(t)] - 2tu_k(t)\right)\right]dt$$

$$= x^2 + x^4 + \frac{x^6}{6} - \frac{4\Gamma(5-\alpha)x^{5-\alpha}}{\Gamma(4-\alpha)\Gamma(6-\alpha)} - \frac{4x^{3-\alpha}}{\Gamma(4-\alpha)} - \frac{12x^{5-\alpha}}{\Gamma(6-\alpha)}$$

$$+ \frac{2x^{4-2\alpha}}{\Gamma(5-2\alpha)} \qquad \dots (41)$$

$$u(x) = u_0(x) + u_2(x) + u_3(x) + \cdots$$

$$u(x) \cong 1 + 3x^2 + \frac{3x^4}{2} + \frac{x^6}{6} - \frac{6x^{3-\alpha}}{\Gamma(4-\alpha)} - \frac{4\Gamma(5-\alpha)x^{5-\alpha}}{\Gamma(4-\alpha)\Gamma(6-\alpha)} - \frac{12x^{5-\alpha}}{\Gamma(6-\alpha)}$$

$$+ \frac{2x^{4-2\alpha}}{\Gamma(5-2\alpha)} \qquad \dots (42)$$

Table 6: Numerical values when ∝=0.9, 0.8, 0.7, 0.5 and 1.0 for Eq. (33)

X	EXACT	HYBRID SOLN (∝=1, n=3)	HYBRID SOLN (∝=0.9, n=3)	HYBRID SOLN (∝=0.8, n=3)	HYBRID SOLN (∝=0.7, n=3)	HYBRID SOLN (∝=0.5, n=3)
0	1	1	1	1	1	1
0.1	1.01005	1.010050167	1.013599648	1.017154936	1.020262319	1.024758595
0.2	1.040811	1.040810667	1.052156535	1.063862959	1.074765635	1.092425809
0.3	1.094174	1.0941715	1.116396376	1.139624459	1.161822003	1.200070174
0.4	1.17351	1.173482667	1.209316204	1.247530721	1.284125866	1.349679274
0.5	1.284017	1.283854167	1.335532699	1.393326398	1.447468463	1.546914629
0.6	1.433276	1.432576	1.501076956	1.585421201	1.660593066	1.800864524
0.7	1.63206	1.629658167	1.713282934	1.835204259	1.935411256	2.124157544
0.8	1.895481	1.888490667	1.980712841	2.157551463	2.287438401	2.533303148
0.9	2.24456	2.2266235	2.313101285	2.571490095	2.736400755	3.049211092
1	2.708333	2.666666667	2.721310515	3.101005557	3.306993531	3.697867688

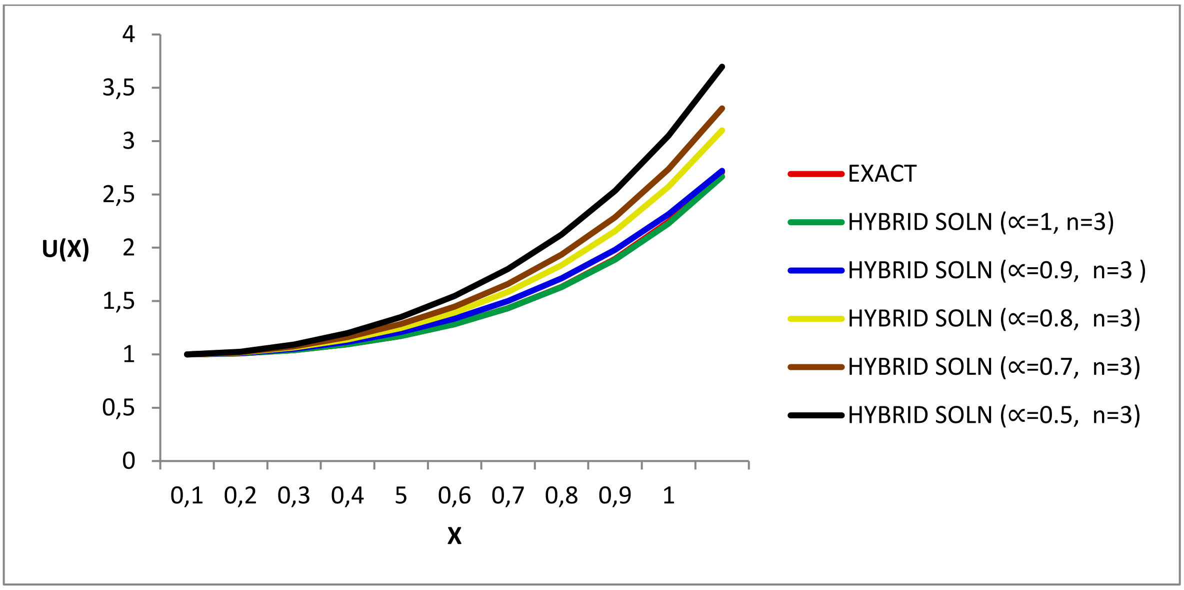

Figure 5: Comparison of approximate solutions obtained by the Hybrid method with exact solution when α=1, 0.9, 0.8, 0.7 and 0.5 of example 1

Discussion of Results

There are two main goals that we aimed for this work. The first is to employ the powerful Hybrid method to investigate linear and nonlinear ordinary differential equations of integer and non-integer order. The second is to show the power of this method and its significant features. The two goals are achieved. It is obvious that the method gives rapid convergent successive approximations without any restrictive assumptions or transformation that may change the physical behavior of the problem. The Hybrid method reduces the size of calculations by not requiring the tedious Adomian polynomials and the Lagrange's multiplier; hence the iteration is direct and straightforward. *Figure* 1: shows the comparison of approximate solutions obtained by the Hybrid method, VIM, ADM and NIM with exact solution for example 1 for a equal to 1, we observed that the Hybrid method with just three (3) iteration is in good agreement with the exact solution and by far better than the NIM with five (5) iteration, ADM with five (5) iteration and VIM with six (6) iteration.

Figure 2 is a graph of the logistic fractional differential equation solutions of Example 1 using the Hybrid method with different a values. It was observed that the effect of the fractional-order a in the solution of the logistic differential equation is that the lower the value of a the faster the growth of the population.

It is remarkable to note that the graph of the Hybrid method and the exact solution coincide when **$\alpha=1$ as** depicted in *Figure 4*, the comparison shows that as $\alpha \rightarrow 1$, the approximate solution tends to $\tanh(x)$, which is the exact solution of the problem. The method facilitates the computational work and gives better results than the ADM and ODM as shown in *Figure 3*. However, for concrete problems, such as nonlinear fractional order integral and integro-differential equations problem, a good number of iterations are needed to get a reasonable accuracy level. This will be shown in our forthcoming work.

Conclusion

An effective hybrid method combining the Adomian Decomposition Method (ADM) and the Variational Iteration Method (VIM) has been successfully developed for solving both linear and nonlinear ordinary differential equations (ODEs), including their fractional-order counterparts. The proposed technique demonstrates robust accuracy and convergence, offering a reliable analytic approximation framework that outperforms several established methods such as ADM, ODM, VIM, and NIM. Application to the fractional logistic differential equation reveals that lower values of the fractional order α\alpha correspond to more rapid initial population growth, providing valuable modeling insights in demographic studies. These findings validate the method's effectiveness and underscore its potential for broader applications. Future research should explore its extension to nonlinear fractional integral equations and other complex systems in applied science, further enhancing its utility and impact in mathematical modeling.

References

Bhalekar, S., and Daftardar-gejji, V. (2012). Solving Fractional-Order Logistic Equation Using a

New Iterative Method. *International Journal of Differential Equations* Volume 2012, Article ID 975829, 12 pages. *2012*. https://doi.org/10.1155/2012/975829

Hemeda, A. A. (2018). A Friendly Iterative Technique for Solving Nonlinear Integro-Differential and Systems of Nonlinear Integro-Differential Equations. *International Journal of Computational Methods*, *15*(1), 1–15. https://doi.org/10.1142/S0219876218500160

Rashidi, M.M., Rabiei, F. S., and Abbasbandy, S. (2020). A Review : Differential Transform Method. *Int. Journal. of Applied Mechanics and Engineering*, *25*(2), 122 129.https://doi.org/10.2478/ijame-2020-0024

Momani, S., and Odibat, Z. (2007). Homotopy perturbation method for nonlinear partial differential equations of fractional order. *Physics Letters, Section A: General, Atomic and Solid State Physics*, *365*(5–6), 345–350. https://doi.org/10.1016/j.physleta.2007.01.046

Odibat, Z. (2019). An optimized decomposition method for nonlinear ordinary and partial differential equations. *Physica A* .https://doi.org/10.1016/j.physa.2019.123323

Tate, S., and Dinde, H. T. (2019). A New Modification of Adomian Decomposition Method for Nonlinear Fractional-Order Volterra Integro-Differential Equations. *World Journal of Modelling and Simulation 15*(1), 33–41.

Wazwaz, A. M. (2009). The variational iteration method for analytic treatment for linear and nonlinear ODEs. *Applied Mathematics and Computation*, *212*(1), 120–134. https://doi.org/10.1016/j.amc.2009.02.003

Ziane, D., and Cherif, M. H. (2018). Variational iteration transform method for fractional differential equations. *Journal of Interdisciplinary Mathematics*, *21*(1), 185–199. https://doi.org/10.1080/09720502.2015.1103001