Asian Journal of
Science, Technology, Engineering, and Art

e-ISSN: 3025-4507
p-ISSN: 3025-5287

Index: Harvard, Boston, Sydney University, Dimensions, Lens, Scilit, Semantic, Google, etc

https://doi.org/10.58578/AJSTEA.v3i3.5453

Unidirectional and Bidirectional Coupling Combination Synchronization of Double-Well Duffing Oscillators and Van der Pol Duffing Oscillators Using Linear State Feedback Control Techniques

A. N. Njah[1], A. M. Lasisi[2*], O. Olabisi[3], A. H. Adeshina[4]
[1]University Of Lagos, Lagos State, Nigeria; [2,4]Ajayi Crowther University, Oyo, Oyo State, Nigeria; [3]Ladoke Akintola University Of Technology, Ogbomoso, Oyo State, Nigeria
am.lasisi@acu.edu.ng

Article Info:

Submitted:	Revised:	Accepted:	Published:
Mar 23, 2025	Apr 7, 2025	Apr 19, 2025	Apr 24, 2025

Abstract

This paper investigates the synchronization of complex dynamical systems, specifically Double-Well Duffing Oscillators and Van der Pol Duffing Oscillators, using unidirectional and bidirectional coupling with linear state feedback control techniques. These oscillators exhibit nonlinear behavior that is widely relevant in engineering and physics, and their coupling mechanisms play crucial roles in applications like signal processing, secure communications, and mechanical systems. In this work, linear state feedback control was employed. The synchronization behavior under both unidirectional and bidirectional coupling schemes was examined. The system was observed by analyzing the stability, convergence rate, and effectiveness of the coupling. MATLAB software was used to validate the feasibility and effectiveness of the linear feedback coupling technique. The results demonstrate that combination synchronization can be achieved through the application of linear state error

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feedback control techniques in either unidirectional coupling or bidirectional coupling.

Keywords: Double-Well Duffing Oscillators, Van der Pol Duffing Oscillators, Linear state feedback, Unidirectional coupling, Bidirectional coupling

Introduction

Synchronization of nonlinear oscillators has been a subject of significant interest in various fields of science and engineering, including physics, biology, and control systems. Among these oscillators, the Double-Well Duffing Oscillator and the Van der Pol Duffing Oscillator serve as fundamental models that describe a wide range of natural and engineered systems exhibiting nonlinear behaviors. These oscillators exhibit complex dynamics, including periodic, quasi-periodic, and chaotic motion, making their synchronization a challenging and intriguing problem[1,2,3]. The ability to achieve synchronization in these systems has profound implications for signal processing, secure communication, mechanical system stabilization, and neural network modeling. Synchronization in nonlinear systems has garnered significant interest due to its wide-ranging applications, including secure communication, neurophysiological signal analysis, and energy harvesting systems. Among various nonlinear systems, Duffing and Van der Pol oscillators are prominent for their chaotic and periodic behavior, which can be controlled and synchronized under specific coupling and control schemes. Synchronizing Double-Well Duffing and Van der Pol Duffing oscillators is particularly challenging due to the coexistence of periodic, quasiperiodic, and chaotic states in these systems. By applying linear state feedback control [4,5,6,7]. This work aims to achieve effective synchronization between these oscillators under both unidirectional and bidirectional coupling conditions.

Mathematical Model of the Oscillators

Double-Well Duffing Oscillator

The dynamics of the Double-Well Duffing Oscillator can be described by [8]:

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -\beta x_2 + \frac{x_1}{2} - \frac{x_1^3}{2} + F\cos(\omega t) \qquad (1)$$

$$\dot{y}_1 = y_2\,,$$

$$\dot{y}_2 = -\beta y_2 + \frac{y_1}{2} - \frac{y_1^3}{2} + F\cos(\omega t) + k(x_1 - y_1 - z_1 \quad (2)$$

$$\dot{z}_1 = z_2,$$

$$\dot{z}_2 = -\beta z_2 + \frac{z_1}{2} - \frac{z_1^3}{2} + F\cos(\omega t) + k(x_1 - y_1 - z_1) \quad (3)$$

Equation (1) represents the drive system while equations (2) and (3) represent the response systems.

With $k = 0$, $F = 0.095$, $\omega_1 = \omega_2 = \omega_3 = 0.79$, and β =0.1, equations (1), (2) and (3) describe free running systems exhibiting cross-well chaos when a strong periodic forcing is applied.

Coupling two drive systems and one response system [9,10]

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -\beta x_2 + \frac{x_1}{2} - \frac{x_1^3}{2} + F\cos(\omega t) \quad (4)$$

$$\dot{y}_1 = y_2,$$

$$\dot{y}_2 = -\beta y_2 + \frac{y_1}{2} - \frac{y_1^3}{2} + F\cos(\omega t) \quad (5)$$

$$\dot{z}_1 = z_2,$$

$$\dot{z}_2 = -\beta z_2 + \frac{z_1}{2} - \frac{z_1^3}{2} + F\cos(\omega t) + k(x_1 + y_1 - z_1) \quad (6)$$

Equations (4) and (5) represent the drive systems and equation (6) represents the response system,

Where x_1, y_1 and z_1 represent displacement from the equilibrium position, β denotes the damping constant. F and k respectively stand for the amplitude of external forcing and coupling parameter that controls the strength of linear feedback into the response systems.

Van der Pol Duffing Oscillator

The Van der Pol Duffing Oscillator is governed by:

$$\dot{x}_1 = x_2 \quad ,$$

$$\dot{x}_2 = \mu(1 - x_1^2)x_2 - \alpha x_1 - \beta x_1^3 + F\cos(\omega t) \quad (7)$$

$$\dot{y}_1 = y_2,$$
$$\dot{y}_2 = \mu(1 - y_1^2)y_2 - \alpha y_1 - \beta y_1^3 + F\cos(\omega t) + k(x_1 - y_1 - z_1) \quad (8)$$
$$\dot{z}_1 = z_2,$$
$$\dot{z}_2 = \mu(1 - z_1^2)z_2 - \alpha z_1 - \beta z_1^3 + F\cos(\omega t) + k(x_1 - y_1 - z_1) \quad (9)$$

Equation (7) represents the drive system while equations (8) and (9) represent the response systems.

With $k = 0$, $F = 0.095$, $\omega_1 = \omega_2 = \omega_3 = 0.79$, β =0.5, α =-0.5 and $\mu = 0.1$ equations (7), (8) and (9) describe free running systems exhibiting cross-well chaos when a strong periodic forcing is applied as shown in Figure 1.

Coupling of two drive systems with one response system [11,12,13].

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = \mu(1 - x_1^2)x_2 - \alpha x_1 - \beta x_1^3 + F\cos(\omega t) \quad (10)$$
$$\dot{y}_1 = y_2,$$
$$\dot{y}_2 = \mu(1 - y_1^2)y_2 - \alpha y_1 - \beta y_1^3 + F\cos(\omega t) \quad (11)$$
$$\dot{z}_1 = z_2,$$
$$\dot{z}_2 = \mu(1 - z_1^2)z_2 - \alpha z_1 - \beta z_1^3 + F\cos(\omega t) + k(x_1 + y_1 - z_1) \quad (12)$$

Equations (10) and (11) represent the drive systems and equation (12) represents the response system which is unidirectionally coupled.

Where x_1, y_1 and z_1 represent displacement from the equilibrium position, μ denotes the damping constant. α and β represent the potential parameter, F and k respectively stand for the amplitude of external forcing and coupling parameter that controls the strength of linear feedback into the response systems [14.15.16].

Numerical Simulations and Results

All the results presented here were computed with the following parameter settings for each oscillator. Numerical solutions were obtained using a fouth-order Runge-Kutta routine on MATLAB software.

Unidirectional Combination Synchronization

Double-Well Duffing Oscillator

For the synchronization of double well Duffing oscillator, the parameters of the systems are fixed as follows: $\beta = 0.1$, $F = 0.095$, and $\omega = 0.79$. The initial conditions for the drive and response systems are freely chosen as $(x_1, \dot{x}_1, y_1, \dot{y}_1, z_1, \dot{z}_1)$ = (-2.2, 2.3, -1.2, 1.3, 0.2, 2.9).

For the case of one drive and two response systems, the systems of equation (4) to (6) are solved using MATLAB software to obtain the results displayed on Figure 2.

Similarly for the case of two drives and one response systems, the systems of equations (7) to (9) are solved to obtain the result displayed on Figure 3.

Van der Pol-Duffing Oscillators

For the synchronization of the van der Pol Duffing oscillator, the parameters are fixed as follows:

$\beta = 0.5$, $F = 0.095$, and $\omega = 0.79, \mu = 0.1, \alpha = -0.5$. The initial conditions for the drive and response systems are freely chosen as $(x_1, \dot{x}_1, y_1, \dot{y}_1, z_1, \dot{z}_1)$ = (-2.2, 2.0, -1.2, 0, 0.2, 2)

For the case of one drive and two response systems, the systems of equation (10) to (11) are solved using MATLAB software to obtain the results displayed on Figure 4.

Similarly for the case of two drives and one response systems, the systems of equations (11) to (12) are solved to obtain the result displayed on Figure 5.

Bidirectional Combination Synchronization

Double-Well Duffing Oscillator

Bidirectional Combination Coupling Of Van Der Pol-Duffing Oscillator

Coupling one drive system and two response systems

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = \mu(1 - x_1^2)x_2 - \alpha x_1 - \beta x_1^3 + F\cos(\omega t) + k(z_1 + y_1 - x_1) \quad (12)$$

$$\dot{y}_1 = y_2,$$

$$\dot{y}_2 = \mu(1 - y_1^2)y_2 - \alpha y_1 - \beta y_1^3 + F\cos(\omega t) + k(x_1 - y_1 - z_1) \quad (13)$$

$$\dot{z}_1 = z_2,$$

$$\dot{z}_2 = \mu(1 - z_1^2)z_2 - \alpha z_1 - \beta z_1^3 + F\cos(\omega t) + k(x_1 - y_1 - z_1) \quad (14)$$

Equation (3.23) represents the drive system while equations (3.24) and (3.25) represent the response systems. Where x_1, y_1 and z_1 represent displacement from the equilibrium position, μ denotes the damping constant. α and β represent the potential parameters, F and k respectively stand for the amplitude of external forcing and coupling parameter that controls the strength of linear feedback into the response systems.

Coupling two drive systems with one response system [17,18,19,20]

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = \mu(1 - x_1^2)x_2 - \alpha x_1 - \beta x_1^3 + F\cos(\omega t) + k(z_1 - y_1 - x_1) \quad (15)$$

$$\dot{y}_1 = y_2,$$

$$\dot{y}_2 = \mu(1 - y_1^2)y_2 - \alpha y_1 - \beta y_1^3 + F\cos(\omega t) + k(z_1 - y_1 - z_1) \quad (16)$$

$$\dot{z}_1 = z_2,$$

$$\dot{z}_2 = \mu(1 - z_1^2)z_2 - \alpha z_1 - \beta z_1^3 + F\cos(\omega t) + g(x_1 + y_1 - z_1) \quad (17)$$

Equations (3.26) and (3.27) represent the drive systems and equation (3.28) represents the response system.

Where x_1, y_1 and z_1 represent displacement from the equilibrium position, μ denotes the damping constant. α and β represent the potential parameters, F stand for the amplitude of external forcing while k and g respectively stand for the drive and response coupling parameter that controls the strength of linear feedback into the drive-response systems.

For the synchronization of double well Duffing oscillator, the parameters of the systems are fixed as follows: $\beta = 0.1$, $F = 0.095$, and $\omega = 0.79$. The initial conditions for the drive and response systems are freely chosen as $(x_1, \dot{x}_1, y_1, \dot{y}_1, z_1, \dot{z}_1)$ = (-2.2, 2, -1.2, 0, 0.2, 1).

For the case of one drive and two response systems, the systems of equation (12) to (14) are solved using MATLAB software to obtain the results displayed on Figure 7. Similarly for the case of two drives and one response systems, the systems of equations (15) to (17) are solved to obtain the result displayed on Figure 8.

Van der Pol-Duffing Oscillator

For the synchronization of van der Pol Duffing oscillator, the parameters are fixed as follows:

$\beta = 0.5, F = 0.095$, and $\omega = 0.79, \mu = 0.1, \alpha = -0.5$.

The initial conditions for the drive and response systems are freely chosen as $(x_1, \dot{x}_1, y_1, \dot{y}_1, z_1, \dot{z}_1)$= (-2.2, 1.3, -1.2, 0, 0.2, 1).

For the case of one drive and two response systems, the systems of equation (12) to (14) are solved using MATLAB software to obtain the results displayed on Figure 6.

Similarly for the case of two drives and one response systems, the systems of equations (15) to (17) are solved to obtain the result displayed on Figure 7.

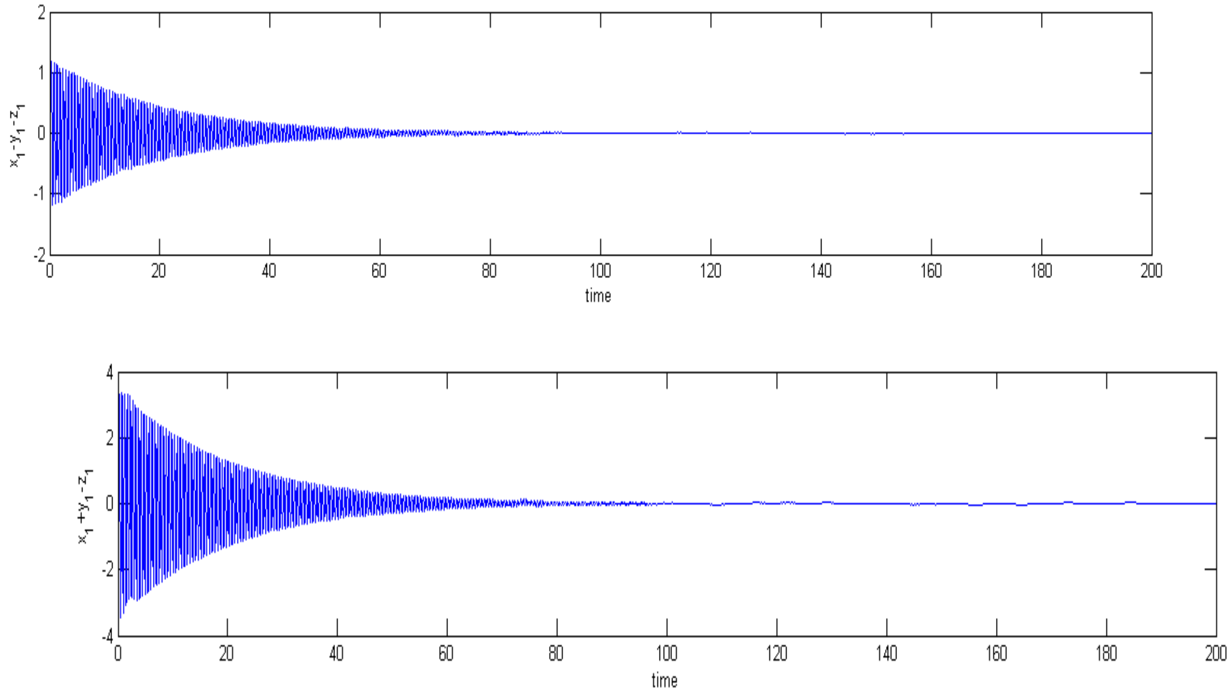

Figure 1 (a) and (b): Time series of error dynamics $(x_i - y_i - z_i \; i = 1,2 \;)$ for the unidirectional combination synchronization of DDO systems with one drive and two response systems with *k* value of 70

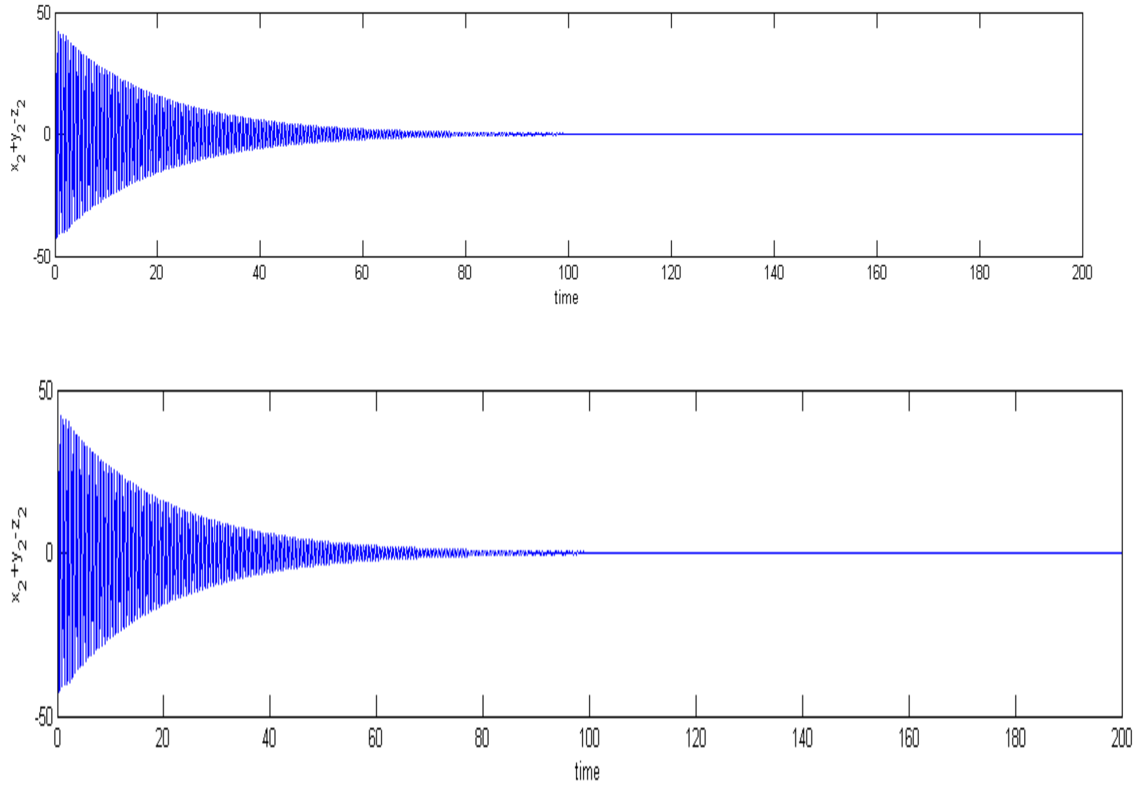

Figure 2 (a) and (b): Time series of error dynamics ($x_i + y_i - z_i, i = 1,2$) for the unidirectional combination synchronization of the DDO systems with two [illegible]

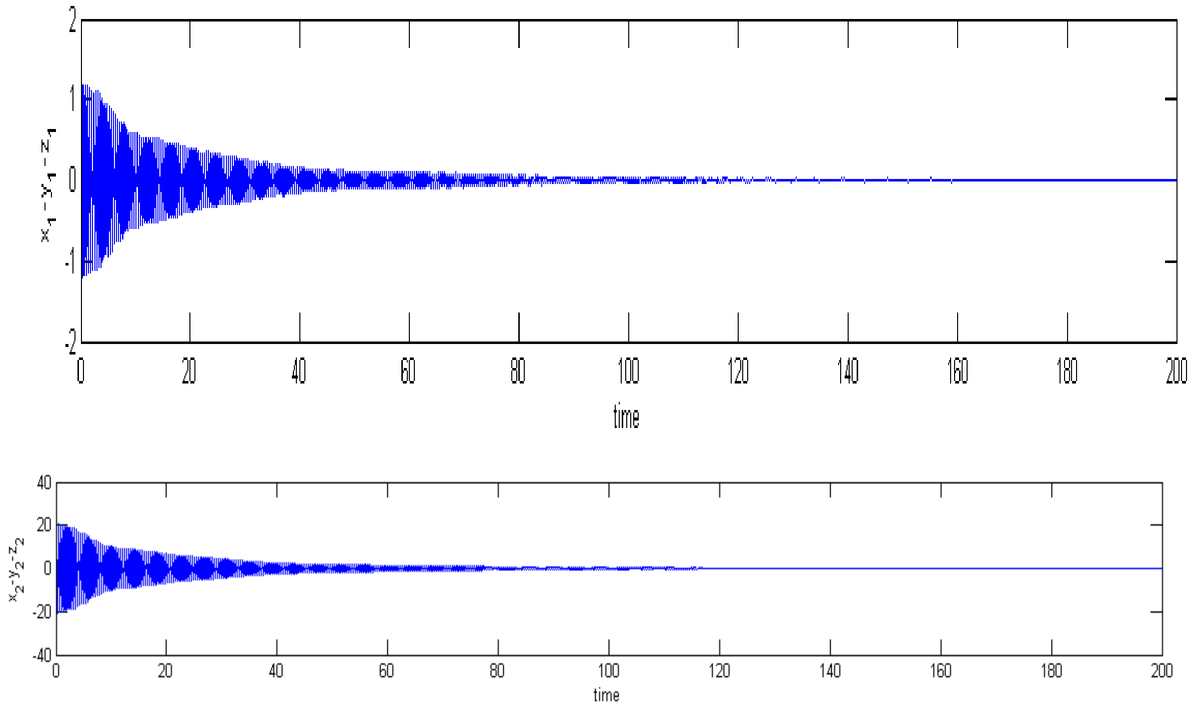

Figure 3 (a) and (b): Time series of error dynamics ($x_i - y_i - z_i$ $i = 1,2$) for the unidirectional combination synchronization of van der Pol systems with one drive and two response systems for k value of 150

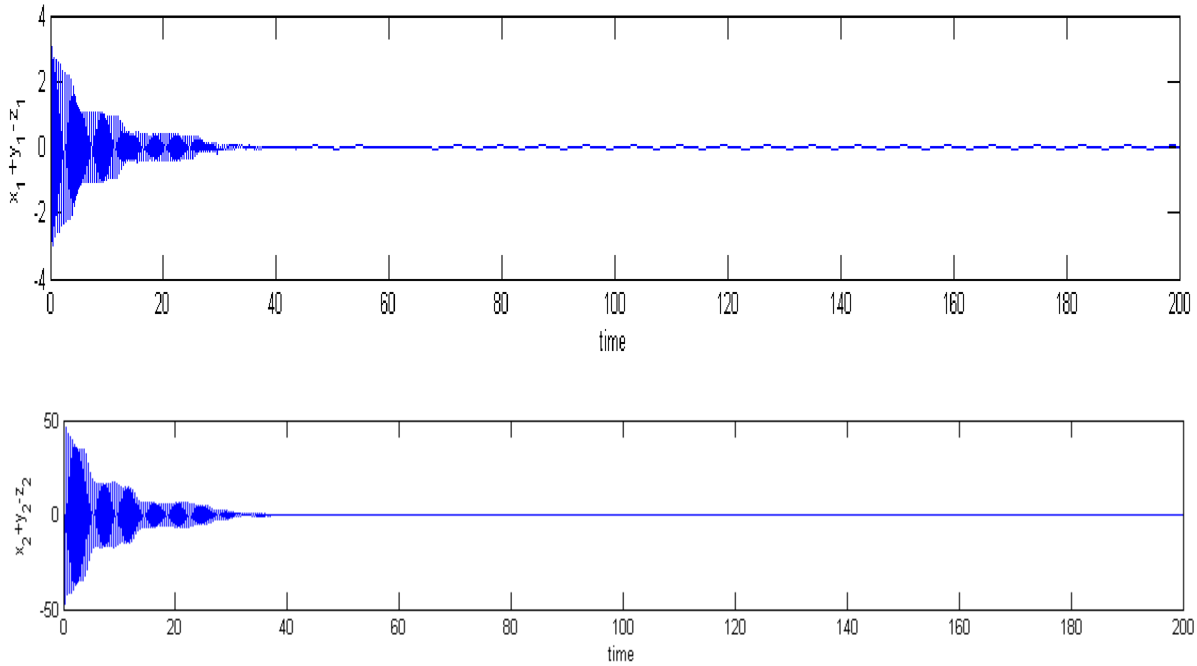

Figure 4 (a) and (b): Time series of error dynamics $(x_i + y_i - z_i \; i = 1,2 \;)$ for the unidirectional combination synchronization of van der Pol systems with two drives and one response systems for k value of 250

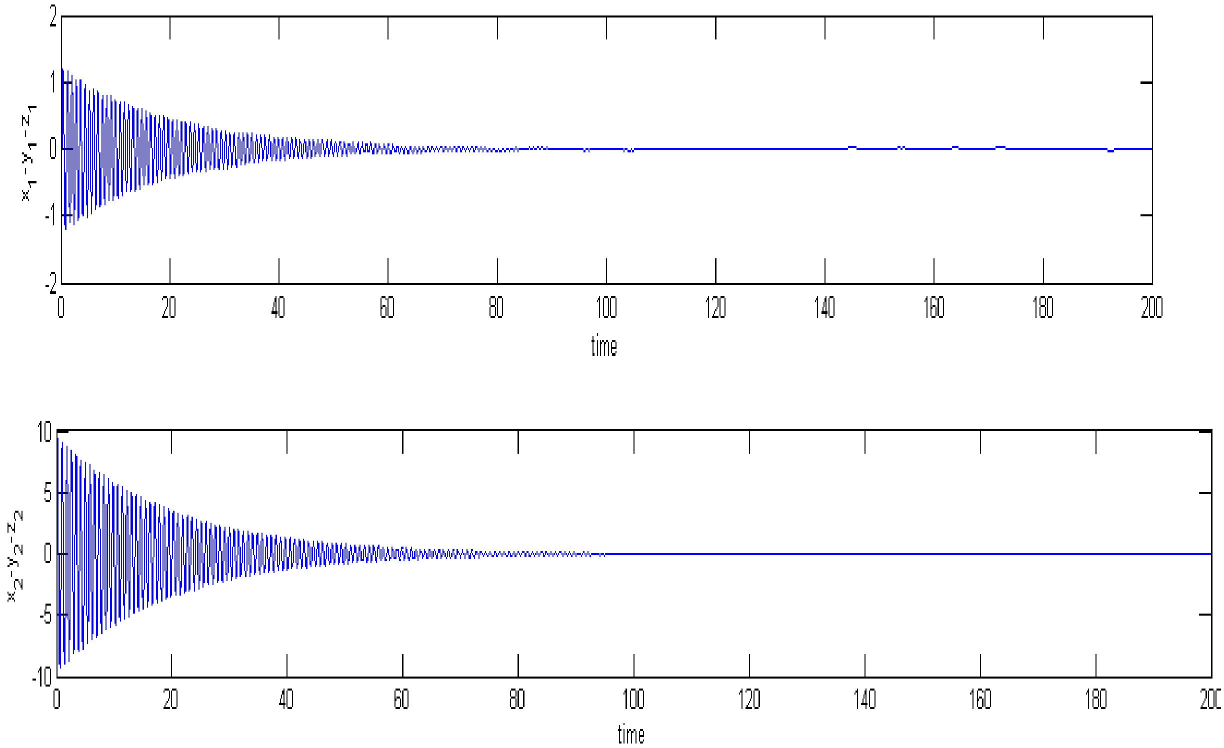

Figure 5 (a) and (b): Time series of error dynamics $(x_i - y_i - z_i \; i = 1,2 \;)$ for the Bidirectional combination synchronization of DDO systems with one drive and two response systems for k value of 20

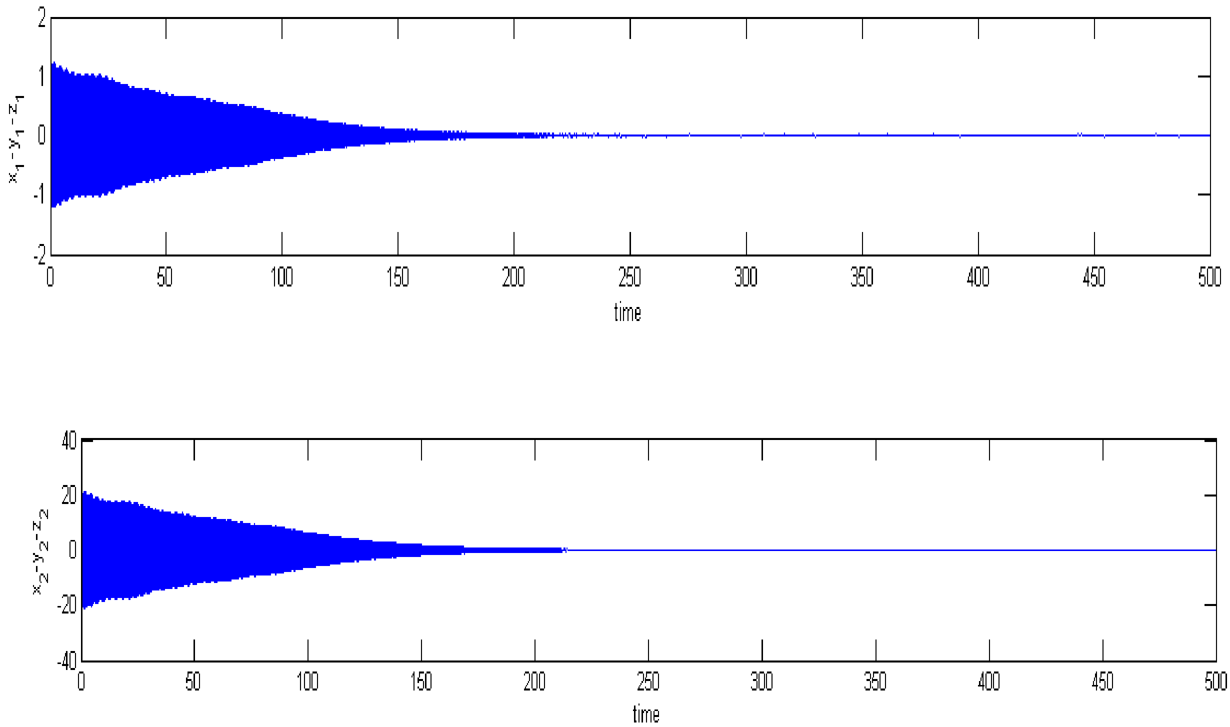

Figure 6 (a) and (b): Time series of error dynamics ($x_i - y_i - z_i$ $i = 1,2$) for the Bidirectional combination Synchronization of van der Pol systems with one drive and two response systems for k value of 100

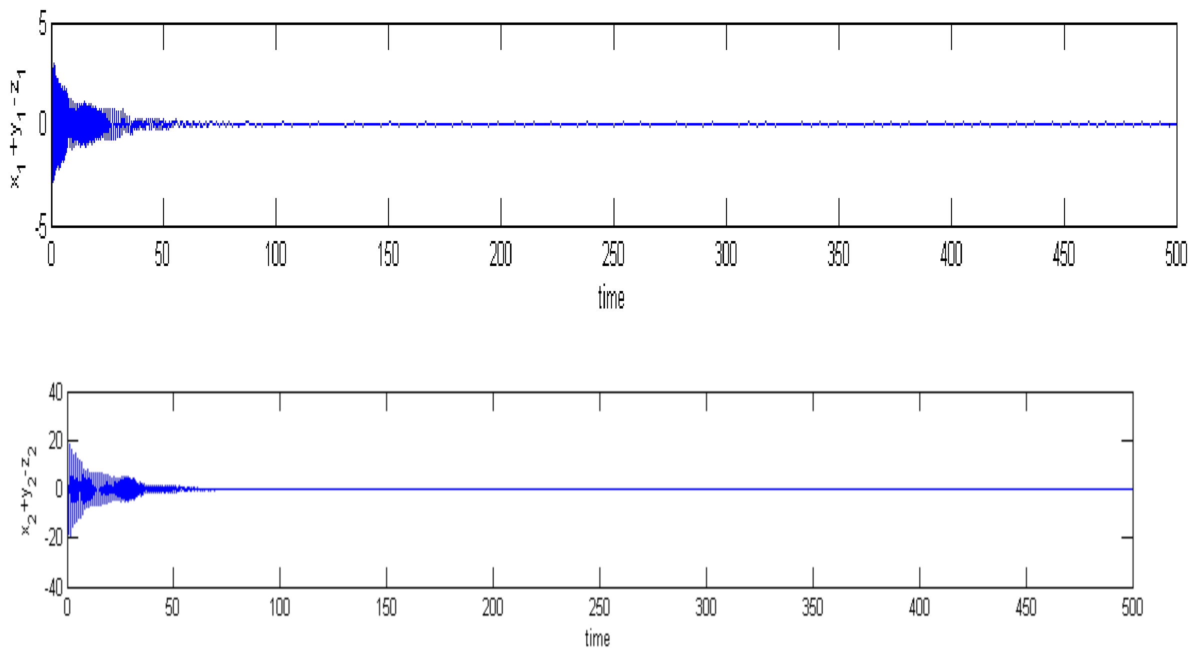

Figure 7 (a) and (b): Time series of error dynamics ($x_i + y_i - z_i$ $i = 1,2$) for the Bidirectional combination synchronization of van der Pol systems with two drives and one response systems for value of 3 and g value of 35

Conclusion

This study demonstrates that linear state feedback control is effective in synchronizing Double-Well Duffing and Van der Pol Duffing oscillators under both unidirectional and bidirectional coupling schemes. Bidirectional coupling provides superior synchronization performance, while unidirectional coupling may be preferred in systems with directional data flow requirements. Numerical simulations are conducted to illustrate the validity and feasibility of the theoretical analysis. Combination synchronization between two drive systems and one response system has obvious advantages over synchronization between one drive system and one response system. Bidirectionally coupled oscillators achieve synchronization faster with less coupling strength than the unidirectionally coupled oscillators.

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