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Derivation of Cumulative Residual Entropy for a Modified Skewed Distribution

M. B. W. Nkombou[1], I. J. David[2], E. N. Okeke[3]
Federal University Wukari, Taraba State, Nigeria
bobnkombou@gmail.com; davidij@fuwukari.edu.ng

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Abstract

In this paper, the derivation of cumulative residual entropy which makes use of a probability distribution survival function for the derivation is presented. The entropy is derived for of a distribution introduced by Nkombou *et al.* (2025) called DUS Skew Student-t (DUSSS-*t*) Distribution. Specifically, the entropy derived is the Cumulative Residual Renyi Entropy (CRRE). The final result shows it is possible that other cumulative residual entropy for the DUSSS-*t* can be estimated following the same approach in this paper.

Keywords: Transformation, Entropy, Cumulative Residual, Skew Distribution, Reliability Function

INTRODUCTION

Entropy measure was introduced by Shannon (1948) for measuring information uncertainty in the field of information theory. On the average, the amount of information required to explain a random variable, q, is a measure of entropy for that specific random variable, q. It is eminent in the literature on entropy that many modification or introduction of entropies other than the Shannon entropy (SE) has been done. The Renyi entropy (RE) is one of such modification of the SE by Renyi (1961). According to Renyi (1961), entropy of a probability distribution (PD) is not just a measure of uncertainty but also a measure of the quantity of information. It is a generalization of entropies like SE, Hartley entropy, collision entropy, and min-entropy. The quantity of information at order z concerning a continuous random variable q which is gotten from the observation of an event E is defined as (David *et al.*, 2024a and 2024b, Mathew *et al.*, 2024, Eghwerido *et al.*, 2020, and Nkombou *et al.*, 2025):

$$R_c = \frac{1}{1-z} log[E_z(\alpha)], \tag{1}$$

where, $E_z(\alpha) = \int_{-\infty}^{+\infty} f(q,\alpha)^z dq$ and $z > 0$, $z \neq 0$ and $f(q,\alpha)$ is the Probability Density Function, PDF, of a specified distribution.

The RE has some drawbacks which includes estimated RE being negative, not having the required properties of information measure, and other limitations. These shortcomings have led researchers to develop an alternative to RE by generalizing the RE through the substitution of the PDF in RE with the survival function (SF) of the distribution. Rao *et al.* (2004) started this generalization when they proposed the generalized version of Shannon's entropy. Zardasht (2022) defined the Cumulative Residual RE (CRRE) of order z (and its dynamic version, DCRRE, by extending it to the residual lifetime variable) as follows:

$$CR_z = \frac{1}{1-z} log[D_z(\alpha)],\ z > 0,\ z \neq 1 \tag{2}$$

where, $D_z(\alpha) = \int_{-\infty}^{+\infty} S(q,\alpha)^z dq$ and $S(q,\alpha)$ is the SF of a specified distribution. Derivation of the CRRE for different existing PDs is not new and can be seen in the work of AI-Babtain *et al.* (2021) and lots more.

METHODS

This section presents the theoretical derivation of CRRE for a modified Skew Student-*t* (SS-*t*) distribution through DUS transformation.

CRRE Derivation for DUSSS-*t* Distribution

Let q be an independent and identically distributed (*i.i.d.*) random variable, *r.v.* The *r.v.* q is said to follow a DUSSS-*t* distribution if it has a SF given as (Nkombou, 2025);

$$S(q,\alpha)=\frac{1}{e-1}\left[e-e^{\frac{1}{2}(1+\frac{q}{\sqrt{\alpha+q^2}})}\right], -\infty<q<\infty, \alpha>0 \tag{3}$$

Lemma: The CRRE for a DUSSS*t* distribution is given as:

$$CR_z=\frac{1}{1-z}log\left[\begin{cases} 0 & \text{for} \quad n\ odd \\ C_{i,h,k,t,n}\dfrac{\Gamma(\frac{h+1}{2})\Gamma(\frac{h-1}{2})}{\Gamma(\frac{h}{2})} & \text{for} \quad n\ even \end{cases}\right]$$

where, $\Psi_{i,h,k,t,n}=\sum_{i,k=0}^{\infty}\sum_{h=0}^{z}\sum_{t=0}^{k}\sum_{n=0}^{i}\binom{z}{h}\binom{k}{t}\binom{i}{h}\frac{h^k e^{z-h}}{k!i!}\frac{(-1)^{t+h}}{(e-1)^{z+k}}\left(\frac{k-t}{2}\right)^i\alpha^{\frac{1}{2}}$.

RESULTS

Proof: By definition, the CRRE measure is given as:

$$CR_z=\frac{1}{1-z}log[D_z(\alpha)],\ z>0,\ z\neq 1$$

with, $D_z(\alpha)$ given by Equation (4), the survival function of the DUSSS-*t* distribution is given by Equation (3). The derivation of $D_z(\alpha)$ is given as:

$$D_z(\alpha)=\int_{-\infty}^{+\infty}\left[\frac{1}{e-1}\left[e-e^{\frac{1}{e-1}\left(e^{\frac{1}{2}\left(1+\frac{q}{\sqrt{\alpha+q^2}}\right)}-1\right)}\right]\right]^z dq$$

$$=\left(\frac{1}{e-1}\right)^z\int_{-\infty}^{+\infty}\left[e-e^{\frac{1}{e-1}\left(e^{\frac{1}{2}\left(1+\frac{q}{\sqrt{\alpha+q^2}}\right)}-1\right)}\right]^z dq$$

$$=\left(\frac{1}{e-1}\right)^z\int_{-\infty}^{+\infty}\sum_{h=0}^{z}\binom{z}{h}e^{z-h}(-1)^h\left[e^{\frac{1}{e-1}\left(e^{\frac{1}{2}\left(1+\frac{q}{\sqrt{\alpha+q^2}}\right)}-1\right)}\right]^h dq$$

$$=\Big(\frac{1}{e-1}\Big)^{z}\sum_{h=0}^{z}\binom{z}{h}e^{z-h}(-1)^{h}\int_{-\infty}^{+\infty}e^{\frac{h}{e-1}\Big(e^{\frac{1}{2}\Big(1+\frac{q}{\sqrt{\alpha+q^{2}}}\Big)}-1\Big)}dq$$

$$=\Big(\frac{1}{e-1}\Big)^{z}\sum_{h=0}^{z}\binom{z}{h}e^{z-h}(-1)^{h}\int_{-\infty}^{+\infty}\sum_{h=0}^{\infty}\frac{1}{k!}\Big[\frac{h}{e-1}e^{\frac{1}{2}\Big(1+\frac{q}{\sqrt{\alpha+q^{2}}}\Big)}-1\Big]^{k}dq$$

$$=\Big(\frac{1}{e-1}\Big)^{z}\sum_{h=0}^{\infty}\binom{z}{h}e^{z-h}(-1)^{h}\sum_{k=0}^{\infty}\frac{1}{k!}\Big(\frac{h}{e-1}\Big)^{k}\int_{-\infty}^{+\infty}\Big[\frac{h}{e-1}e^{\frac{1}{2}\Big(1+\frac{q}{\sqrt{\alpha+q^{2}}}\Big)}-1\Big]^{k}dq$$

$$=\Big(\frac{1}{e-1}\Big)^{z}\sum_{h=0}^{z}\sum_{k=0}^{\infty}\binom{z}{h}\frac{e^{z-h}}{k!}\left(\frac{h}{e-1}\right)^{k-t}\int_{-\infty}^{+\infty}(-1)^{h}\sum_{t=0}^{k}\Big[e^{\frac{1}{2}\Big(1+\frac{q}{\sqrt{\alpha+q^{2}}}\Big)}\Big]^{k-t}(-1)^{t}\binom{k}{t}dq$$

$$=\sum_{h=0}^{c}\sum_{k=0}^{\infty}\sum_{t=0}^{k}\binom{c}{h}\binom{k}{t}\frac{h^{k}e^{c-h}}{k!}\frac{(-1)^{t+h}}{(e-1)^{c+k}}\int_{-\infty}^{+\infty}\sum_{i=0}^{\infty}\frac{1}{i!}\left(\frac{k-t}{2}\right)^{i}\left(1+\frac{q}{\sqrt{\alpha+q^{2}}}\right)^{i}dq$$

$$=\sum_{i,k=0}^{\infty}\sum_{h=0}^{z}\sum_{t=0}^{k}\sum_{h=0}^{i}\binom{z}{h}\binom{k}{t}\binom{i}{h}\frac{h^{k}e^{z-h}}{k!i!}\frac{(-1)^{t+h}}{(e-1)^{z+k}}\Big(\frac{k-t}{2}\Big)^{i}\int_{-\infty}^{+\infty}q^{h}(\alpha+q^{2})^{-\frac{h}{2}}dq.$$

Based on the concept of Taboga (2017) then,

$$\int_{-\infty}^{+\infty}q^{h}f(q,\alpha)dq\;=\big[1-(-1)^{h}\big]\int_{0}^{+\infty}q^{h}f(q,\alpha)dq \tag{4}$$

where, $f(q,\alpha)$ is considered as the PDF of a given probability distribution.

Let $p=\frac{q^{2}}{\alpha}$ and $q=(p\alpha)^{\frac{1}{2}}$, differentiating the expression w.r.t *q,* it yields $dq=\frac{\alpha.dp}{2(p\alpha)^{\frac{1}{2}}}$.

By substitution into Equation (4), the integral part of the expression of $D_z(\alpha)$ is expressed as:

$$\int_{-\infty}^{+\infty}q^{h}(\alpha+q^{2})^{-\frac{h}{2}}dq=\big[1-(-1)^{h}\big]\int_{0}^{+\infty}(p\alpha)^{\frac{h}{2}}(1+\frac{q^{2}}{\alpha})^{-\frac{h}{2}}\frac{\alpha dp}{2(p\alpha)^{\frac{1}{2}}}$$

$$=\big[1-(-1)^{h}\big]\frac{\alpha^{\frac{1}{2}}}{2}\int_{0}^{+\infty}p^{\frac{1}{2}(h-1)}(1+p)^{\frac{-h}{2}}dp \tag{5}$$

Using the Gamma function expression given as:

$$\int_0^{+\infty}\frac{q^a}{(1+q)^b}dq=\frac{\Gamma(a+1)\Gamma(b-a-1)}{\Gamma(b)} \tag{6}$$

Equation (5) becomes:

$$\int_{-\infty}^{+\infty}q^h(\alpha+q^2)^{-\frac{h}{2}}dq=\frac{1}{2}\left[1-(-1)^h\right]\alpha^{\frac{1}{2}}\frac{\Gamma(\frac{h-1}{2}+1)\Gamma(\frac{h}{2}-\frac{h-1}{2}-1)}{\Gamma(\frac{h}{2})}$$

$$=\frac{1}{2}\left[1-(-1)^h\right]\alpha^{\frac{1}{2}}\frac{\Gamma(\frac{h+1}{2})\Gamma(\frac{h-1}{2})}{\Gamma(\frac{h}{2})}$$

Finally, $D_z(\alpha)$ is expressed as

$$D_z(\alpha)=\begin{cases}0 & \text{for} \quad n\ odd\\ \Psi_{i,h,k,t,n}\dfrac{\Gamma(\frac{h+1}{2})\Gamma(\frac{h-1}{2})}{\Gamma(\frac{h}{2})} & \text{for} \quad n\ even\end{cases}$$

where, $\Psi_{i,h,k,t,n}=\sum_{i,k=0}^{\infty}\sum_{h=0}^{z}\sum_{t=0}^{k}\sum_{n=0}^{i}\binom{z}{h}\binom{k}{t}\binom{i}{h}\frac{h^k e^{z-h}}{k!i!}\frac{(-1)^{t+h}}{(e-1)^{z+k}}\left(\frac{k-t}{2}\right)^i\alpha^{\frac{1}{2}}.$

Therefore substituting $D_z(\alpha)$ into Equation (3), the CRE of the DUSSS-*t* distribution is obtained and given as:

$$CR_z=\frac{1}{1-z}log\left[\begin{cases}0 & \text{for} \quad n\ odd\\ \Psi_{i,h,k,t,n}\dfrac{\Gamma(\frac{h+1}{2})\Gamma(\frac{h-1}{2})}{\Gamma(\frac{h}{2})} & \text{for} \quad n\ even\end{cases}\right]$$

CONCLUSION

In this article, the derivation of CRRE for DUSSS-*t* distribution has been theoretically presented. It is obvious that other measures of cumulative residual entropies can be derived for the DUSSS-*t* distribution and in the future this will be explored for entropies like Tsalis, Arimoto, etc., by the researchers.

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