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Gromov - Hausdorff Dimension and Measure of Separable Metric Spaces

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Abstract

In this paper, we proved that dimension of Gromov – Hausdorff hyperspace of Riemannian manifold depends on the cardinality of measurable Riemannian manifold.

Keywords: Riemannian Manifold, Separable Space, Topological Dimension, Lebesgue Measure, Hausdorff Measure

Introduction

In this section, we are going to discuss about Gromov - Hausdorff dimension of a separable metric space X. The Gromov – Hausdorff dimension of a metric space need not be an integer (Facundo, and Zhengchao, 2021). Though, it can be used to measure the structure of limit spaces of a sequence of Riemannian manifold. The Gromov – Hausdorff measure is also called Gromov – Lebesgue measure, in fact, if the subset $Y \subset X$ is a Lebesgue measurable space in Gromov – sense, then it is called Gromov – Lebesgue measurable space. So, for differentiable manifolds, the Gromov – Hausdorff measure is considered as Gromov – Lebesgue measure (Marc, 2009). The Gromov – Hausdorff distance of metric space X depends on the nature of metric space or Riemannian manifold X. If the metric space X is induced with zero dimensional Gromov – Hausdorff measure, Gromov – Hausdorff measure of $X = 0, iff X = \emptyset$, Gromov – Hausdorff measure of $X = n$, if and only if X is finite and Gromov – Hausdorff measure of $X > 0$ if and only if X is infinite set, (Fremlin, 2000).

Definition of some important terms

Definition 2.1 (Dong and Gabjin, 2000): Let X be a compact metric space. For $\epsilon > 0$, define $Cov(X, \epsilon)$ as the minimal number of closed ϵ –balls needed to cover X and $Cap(X, \epsilon)$ as the maximal number of disjoint ϵ – balls in X . $Cov(X, \epsilon)$ is called $\epsilon - covering$ and $Cap(X, \epsilon)$ is called $\epsilon - capacity$ of X .

Definition2.2 (Gromov, 1954)**:** A metric space X is totally bounded if for any $\epsilon > 0$, there exists a finite number N_ϵ of $\epsilon - balls$ $\{B_i = B(x_i, \epsilon)\}_{i=1}^{N_\epsilon}$ which covers X.

Definition 2.3 (Morawo, *et.al.,* 2024): A topological space X is finite if $\dim(X) < \infty$.

Definition 2.4 (Morawo, *et.al.,* 2024): A metric space X is complete if there is a sequence x_n in X such that $x_n \to L \in X$.

Definition 2.5 (Monsuru, Ahmadu and Balla, 2020): A function $f: X \to Y$ is continuous, if for $O \in Y, f^{-1}(O) \in X$.

Definition 2.6 (Monsuru and Ahmadu, 2020): A topological space X is everywhere dense in Y if and only if $cl(X) = Y$.

Definition 2.7 (Morawo, *et.al.,* 2024): A topological space X is called a Riemannia manifold, if Riemannian metric is defined on it.

Definition 2.8 (Morawo, *et.al.,* 2024): A topological space of infinite dimensional is called an infinite dimensional topological space.

Definition 2.9 (Willard, 2004): A topological space X is separable if it has a countable everywhere dense subset.

Definition 2.10 (Morawo, *et.al.,* 2024): A function $f: X \to Y$ is called $\varepsilon - isometry$ if $|d_Y(f(x) - f(y)) - d_X(x, y)| < \varepsilon$ for every $x, y \in X$.

Definition 2.11 (Morawo , *et.al.*, 2023): A function $f: X \to Y$ is called $\delta - isometry$ if $|d_Y(f(x) - f(y)) - d_X(x, y)| < \delta$ for every $x, y \in X$.

Definition 2.12 (Shanti and Raisinganian, 1965): A function $f: X \to Y$ is called an $isometry$ if $d_Y(f(x) - f(y)) \leq d_X(x, y)$, for every $x, y \in X$.

Definition 2.13 (Morawo, *et.al.,* 2024): A map $i: X \to Y$ is an isometric, if X and Y are isometry space.

Definition 2.14 (Paige, 2022): A topological space X is Hausdorff if for $A, B \in X, A \cap B = \emptyset$.

Methodology

By (Gromov, 1954) and (Olga, G. M., and Peter, W.M. 1991), let X be a separable metric space and p an arbitrary real number, $0 \leq p < \infty$. Given $\epsilon > 0$, let

$$\mathcal{GH}_{p.\epsilon}(X) = \frac{\pi^{p/2}}{\Gamma(\frac{p}{2}+1)} inf\{\textstyle\sum_{i=1}^{\infty} r_i^p : X = \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i \leq \epsilon\}, \text{ where } B(x_i, r_i)$$

represents the open ball in X with radius r_i centered at $x_i \in X$ and Γ is called a Gamma function defined by

$$\Gamma(\mathrm{t}) = \int_0^{\infty} e^{-s} s^{t-1} ds, \text{ (Zorich, 2004)}$$

Note that if $p \geq 2$ is a positive integer, then the constant

$$\frac{\pi^{p/2}}{\Gamma(\frac{p}{2}+1)} = \frac{1}{p} vol(S^{p-1}) = \frac{\omega_{p-1}}{p}, \text{ which is exactly equal to } \frac{1}{p} - multiple \text{ of}$$

the volume of round $(p-1) - sphere$. Recall that the volume of a ball of radius r in the Euclidean space $\mathbb{R}^p$ is given by $\frac{1}{p} vol(S^{p-1})$.

Definition 3.1 (Dong and Gabjin, 2000)**:** Let X be a separable metric space and p an arbitrary real number, $0 \leq p < \infty$. The $p-$ dimensional Gromov - Hausdorff measure of X is defined as $\mathcal{GH}_p(X) = \lim_{\epsilon \to 0} \mathcal{GH}_{p.\epsilon}(X)$.

The Gromov - Hausdorff measure looks like the Lebesgue measure. In fact, if A is a Lebesgue measurable subset of $\mathbb{R}^n$, the $n-$ dimensional Gromov - Hausdorff measure $\mathcal{GH}_n(A)$ is equal to Lebesgue measure of A (Masaki, 2022).

For smooth manifolds, The Gromov - Hausdorff measure can be consider as generalization of the Lebesgue measure (Facundo, 2012). The basic properties are the followings;

Proposition 3.1 (Dong, and Gabjin, 2000)**:** For the zero dimensional $GH-$ measure, we have the following properties;

a. $\mathcal{GH}_0(X) = 0$, if $X = \emptyset$
b. $\mathcal{GH}_0(X) = n$, if X is finite set of n points
c. $\mathcal{GH}_0(X) = \infty$, if X is an infinite set

Proposition 3.2: (Dong, and Gabjin, 2000) If $p < q$, then $\mathcal{GH}_p(X) \geq \mathcal{GH}_q(X)$; in fact, $p < q$ and $\mathcal{GH}_p(X) < \infty$ imply that $\mathcal{GH}_q(X) = 0$.

4.0 Results and Discussion

Now, we are going to make use of **Proposition 3.1** above to provide proof for the **Theorem 4.1** below. Also, note that a metric space is separable if it contains a countable everywhere dense subset, and since every finite countable space is compact, then we have the result below.

Theorem 4.1: Suppose X is a compact separable metric space. Then $\mathcal{GH}_p(X) = 0$ if and only if for each $\epsilon > 0$, there is a finite decomposition of X as:

$X = O_1 \cup O_2 \cup \ldots \cup O_n, O_i = O(x_i, r_i)$ such that

$$\sum_{i=1}^{n} k_i^p < \epsilon.$$

Proof: Suppose $\mathcal{GH}_p(X) = 0$ and given that there is $\epsilon > 0$. By definition separability of a metric space, there is a countable number of balls $O(x_1, k_i^1), O(x_2, k_2^1), \ldots$ such that

$$X = \bigcup_{i=1}^{\infty} O(x_1, k_i^1) \; and \; \sum_{i=1}^{\infty} {k^1}_i^p < \frac{\epsilon}{2}.$$

It is easy to expand each ball $O(x_i, k_i^1)$ faintly to an open ball $O(x_i, k_i)$ such that

$$k_i^p < k1_i^p + \frac{\epsilon}{2^{i+1}}.$$

Provided that metric X is compact, then there exists finite covers $O_1, O_2, \dots, O_n, B_k = O(x_n, k_n)$ of X. Therefore,

$$\sum_{i=1}^{n} k_i^p < \epsilon. \blacksquare$$

In the next result, we are going to make use of **definition 4.2, Lemma 4.2** below and **Proposition 3.2** above to establish the dimension of Gromov – Hausdorff hyperspace and its convergence in the Gromov sense.

Definition 4.2 (Paige, 2022)**:** The Gromov – Hausdorff dimension of an arbitrary separable metric space X is given by

$$dim_{\mathcal{GH}}(X) = inf\{p \geq 0: \mathcal{GH}_p(X) = 0\} = sup\{p \geq 0: GH_p(X) > 0\}$$

Note that even if $dim_{\mathcal{GH}}(X) = p$, we have $\mathcal{GH}_p(X) = 0, \infty$ or positive real number. In particular, for a smooth Riemannian $n -$ manifold (M^g, g), we have $dim_{\mathcal{GH}}(X) = n$ and $\mathcal{GH}_n(M) = vol(M, g)$.

Lemma 4.2 (Dong and Gabjin, 2000)**:** For a compact metric space X and $\epsilon > 0$, $Cov(X, 2\epsilon) \leq Cap(X, \epsilon)$.

Theorem 4.2: Given positive real number $k, P > 0$ and a natural number $n \in \mathbb{N}$, let (M_i, g_i) be a sequence of Riemannian n – manifold satisfying $Ric(M_i) \geq -(n-1)k$, $diam(M_i) \leq P$. Suppose X_l is the Gromov – Hausdorff hyperspace limit of the sequence (M_i, g_i). Then $dim_{\mathcal{GH}} X_l \leq n$.

Proof: By **proposition 3.2**, it suffices to show that $\mathcal{GH}_n(X_l) < \infty$. Recall that X_l is a compact inner metric space with $diam(X_l) \leq P$. For every $\epsilon > 0$, recall also that $Cov(X, \epsilon)$ denotes the smallest number of closed $\epsilon - balls\ in\ X$ which cover X.

Also, provided that $Cov(X, \epsilon)$ is continuous in the Gromov Hausdorff sense by **Lemma 4.2**, we have

$Cov(X_l, \epsilon) \leq Cov\left(M_i, \epsilon - \frac{3}{\delta_i}\right), \delta_i \to 0 \; as \; i \to \infty$, for i sufficiently large. The volume comparison theorem together with $diam(M_i) \leq P$, implies that $Cov(M_i, \epsilon)$ depends only on $\epsilon, k \; and \; P$. Thus, we have

$$\mathcal{GH}_n(X_l) = \lim_{\epsilon \to 0} \frac{\omega_{n-1}}{n} \epsilon^n Cov(X_l, \epsilon) \leq \lim_{\epsilon \to 0} \left\{ \lim_{i \to \infty} sup \frac{\omega_{n-1}}{n} \epsilon^n cov\left(M_i, \epsilon - \frac{3}{\delta_i}\right)\right\}$$

$$= \lim_{i \to \infty} sup \left(\frac{\omega_{n-1}}{n} \epsilon^n cov\left(M_i, \epsilon - \frac{3}{\delta_i}\right)\right)$$

$$= \lim_{i \to \infty} sup \, \mathcal{GH}_n(M_i) \leq V_k(P) < \infty.$$

By volume comparison theorem, the last inequality follows. Hence,

$GH_n(M_i) = vol(M_i)$, for a compact smooth Riemannian manifold■

In the next result, we are going to make use of **Proposition 4.1 and Lemma 4.2** below to establish that in a Gromov sense, dimension of hyperspace associated with Riemannian n – manifold is finite dimensional.

Proposition 4.1 (Gromov, 1954)**:** For positive real numbers k and D and a natural number $n \in \mathbb{N}$, let (M_i, g_i) be a sequence of Riemannian n – manifold satisfying $Ric(M_i) \geq -(n-1)k,\; diam(M_i) \leq D$. Suppose X_l is the Gromov – Hausdorff hyperspace limit of the sequence (M_i, g_i). Then every point $x \in X_l$ has a tangent cone

$\mathcal{GH} \lim_{r_i}(X, x, r_i) \equiv T_x X_l$

Lemma 4.2 (Facundo, 2021)**:** For any fixed $\epsilon > 0 \; and \; R > \epsilon$, the function $f: X \to N(\epsilon, R, X)$ is almost continuous in the Gromov – Hausdorff distance.

Theorem 4.3: Given positive real numbers $k, v, D > 0$ and a natural number $n \in \mathbb{N}$, let (M_i, g_i) be a sequence of Riemannian n – manifold satisfying

$Ric(M_i) \geq -(n-1)k \, , vol(M_i) \geq v \; and \; diam(M_i) \leq D$. Suppose X_l is the Gromov – Hausdorff hyperspace limit of the sequence (M_i, g_i). Then $dim_{GH} X_l = n$ and the tangent cone $T_x X_l$ at any point $x \in X_l$ has Gromov – Hausdorff dimension n, i.e

$dim_{\mathcal{GH}} T_x X_l = dim_{\mathcal{GH}} \lim_{r \to \infty}(X_l, x, r.d) = n$, for any point $x \in X_l$.

Proof: Since $vol(M_i) \geq v$, one has $dim_{GH}(X_l) \geq n$ and it follows from the **Theorem 4.2** that $dim_{\mathcal{GH}}(X_l) = n$. It remains to prove that $dim_{\mathcal{GH}} T_x X_l = n$. It is known from the volume condition that for each point $x \in X_l, T_x X$ is a metric cone , i.e , metrically cone on tangent space and so it becomes a length space. As in the proof of **Proposition 3.1,** one has $N(\epsilon, R(X_l, rd)) \leq C\left(\frac{R}{\epsilon}\right)^n$ and so $dim_{\mathcal{GH}} T_x X_l \leq n.$ To show the inequality, it is enough to verify that for any $\alpha > 0, \mathcal{GH}_{n-\alpha}(T_x X_l \cap B(R = 1) = \infty,$ where we designate $B(R = 1)$ as the unit ball in the tangent cone $T_x X_l$ centred at x. Then, we claim that:

$$N(\epsilon, R(X_l, rd)) \leq C^1\left(\frac{R}{\epsilon}\right)^n, \text{ for some positive constant } C^1 = C^1(k, v, D, n) = \infty.$$

Note that, for $N = N(\epsilon, R, X_l)$,

$$volB(R) \leq \textstyle\sum_{i=1}^{N} volB(R = 2\epsilon) \leq N.\max\{volB(R = 2\epsilon)\}. \text{ -------------------------(*)}$$

Provided that N is almost continuous by **Lemma 4.2** above, we have

$$N\big(\epsilon, R(X_l, rd)\big) = N\left(\frac{\epsilon}{r}, \frac{R}{r}, (X, d)\right) \geq C_1.N\left(\frac{\epsilon}{r}, \frac{R}{r}, M_i\right) \geq \frac{volB\left(\frac{R}{r}\right)}{\max\left\{volB\left(\frac{2\epsilon}{r}\right)\right\}},$$ from equation (*) above.

Note that,

$$volB\left(\frac{2\epsilon}{r}\right) \leq V_k\left(\frac{2\epsilon}{r}\right) \leq C\left(\frac{R}{r}\right)^n, \text{ for } r >> 1 \text{ adequately large. So,}$$

$$\frac{volB\left(\frac{2\epsilon}{r}\right)}{\max\left\{volB\left(\frac{2\epsilon}{r}\right)\right\}} \geq C_2 \frac{volB\left(\frac{2\epsilon}{r}\right)}{\left(\frac{2\epsilon}{r}\right)^n} \geq C_2 \frac{vol(M)}{V_k(D)} \frac{V_k\left(\frac{R}{r}\right)}{\left(\frac{2\epsilon}{r}\right)^n}, \text{ from relative volume comparison theorem}$$

$$\geq C_2 \frac{v}{V_k(D)} \frac{\left(\frac{R}{r}\right)^n}{\left(\frac{2\epsilon}{r}\right)^n} = C_1\left(\frac{2\epsilon}{r}\right)^n, \text{ where } C_1 = C_1(k, v, D, n).$$

Finally, choose $\sigma - net$ in $B(R = 1)$ so that balls of radius σ are disjoint and $2\sigma - balls$ cover $B(R = 1)$.

Then for any $\alpha > 0$, we have

$$\sum \frac{\omega_{n-1}}{n}\sigma^{n-\alpha} = \frac{\omega_{n-1}}{n}\sigma^{n-\alpha} N(\sigma, 1, T_x X_l) = \frac{\omega_{n-1}}{n}\sigma^{n-\alpha},$$

Hence, $dim_{\mathcal{GH}} T_x X_l \geq n$ by definition, this completes the proof■

Conclusion

In this work, we proved that Dimension of Gromov – Hausdorff hyperspace of Riemannian manifold depends on the cardinality of Riemannian manifold.

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