Asian Journal of
Science, Technology, Engineering, and Art

e-ISSN: 3025-4507
p-ISSN: 3025-5287

Index: Harvard, Boston, Sydney University, Dimensions, Lens, Scilit, Semantic, Google, etc

https://doi.org/10.58578/AJSTEA.v3i2.5062

Relatively Compactness on Some Hyperspaces Associated with Riemannian Manifolds

Monsuru A Morawo
Bells University of Technology, Ota, Ogun State, Nigeria
morawomonsuruajibola@gmail.com

Article Info:

Submitted:	Revised:	Accepted:	Published:
Feb 9, 2025	Feb 22, 2025	Mar 6, 2025	Mar 11, 2025

Abstract

In this paper, we defined relatively compactness on hyperspaces CL(X) and C(X) of Riemannian metric space X and relatively compactness theorem about metric spaces in the Gromov sense. Some classes of Riemannian manifolds as applications were defined.

Keywords: Relatively compactness, Riemannian manifold, Equicontinuous function, Hyperspace, Totally boundedness

INTRODUCTION

In this section, we follow idea of (Knox, K. S. 2013) to discuss relatively compactness theorem on hyperspaces CL(X) and C(X) of Riemannian metric space X. Relatively compactness theorem is of two types, this first is about metric spaces and the other is about Riemannian manifolds as an application. These theorems are very important in studying universal Riemannian geometry. Also, we are going to make use of (Sergio, A. 2011) notions to deal with applications of those notions to Riemannian manifolds.

Some Important Definitions

Definition 2.1(Dong, C., and Gabjin, Y. 2000): Suppose X is a compact metric space and for $\epsilon > 0$, we define $Cov(X, \epsilon)$ as minimal number of closed ϵ –balls needed to cover metric space X and $Cap(X, \epsilon)$ as maximal number of disjoint $\epsilon -$ balls in X . $Cov(X, \epsilon)$ is called $\epsilon - covering$ and $Cap(X, \epsilon)$ is called $\epsilon - capacity$ of X .

Definition2.2(Gromov, M. 1954)**:** Given a metric space X, then it is totally bounded if for any $\epsilon > 0$, there is a finite neighbourhood N_ϵ of $\epsilon - balls$ $\{B_i = B(x_i, \epsilon)\}_{i=1}^{N_\epsilon}$ which cover X.

Definition 2.3(Hanche & Holden, 2010): A topological space X is said to be paracompact, if every open cover of X has a locally finite open refinement.

Definition 2.4(Heinonen, 2001): A topological space X is said to be first countable space, if there is a countable neighborhood base at each of its points.

Definition 2.5(Hemingsen, 1946): A topological space X is finite if $\dim(X) < \infty$.

Definition 2.6(Hendrinkson, 1999): A metric space X is complete if there is a sequence x_n in X such that $x_n \to L \in X$.

Definition 2.7(Herman, 1990): A function $f: X \to Y$ is continuous, if for $O \in Y, f^{-1}(O) \in X$.

Definition 2.8(Khovanskii, 2013): A topological space X is locally compact if every point $x \in X$ has a compact neighborhood of X.

Definition 2.9(Van, 1998): A topological space X is everywhere dense in Y if and only if $cl(X) = Y$.

Definition 2.10(Engelking, R., and Siekluchi, K. 1992): A topological space X is called a Riemannia manifold, if Riemannian metric is defined on it.

Definition 2.11(Engelking, R., and Siekluchi, K. 1992): A topological space of infinite dimensional is called an infinite dimensional topological space.

Definition 2.12(Monsuru & Kiltho, 2020): A topological space X is extremely disconnected if the closure of every open set in X is open.

Definition 2.13(Wallace, A. H. 2007): A family $\mathcal{F} \subseteq Z(X)$ is called cofinal in $Z(X)$ if for any $Z \subseteq Z(X), \exists Z^* \in \mathcal{F}$ such that $Z \subseteq Z^*$.

Definition 2.14(Willard, S. 2004): A metric space X is uniformly bounded, if it is uniformly bounded below and above.

Definition 2.15(Willard, S. 2004): A topological space X is separable if it has a countable everywhere dense subset.

Definition 2.16(Shanti and Raisinganian, 2008): A function $f: X \to Y$ is called $\varepsilon - isometry$ if $|d_Y(f(x) - f(y)) - d_X(x, y)| < \varepsilon$ for every $x, y \in X$.

Definition 2.17(Watson, 1981): A Hausdorff space is hemi – closed, if there exist a countable subfamily $\mathcal{F} \subseteq Z(X)$ that is cofinal in $Z(X)$.

Definition 2.18(Shanti & Raisinganian, 2008): A function $f: X \to Y$ is called $\delta - isometry$ if $|d_Y(f(x) - f(y)) - d_X(x, y)| < \delta$ for every $x, y \in X$.

Definition 2.19(Shanti & Raisinganian, 2008): A function $f: X \to Y$ is called an $isometry$ if $d_Y\big(f(x) - f(y)\big) \leq d_X(x, y)$, for every $x, y \in X$.

Definition 2.20(Shanti & Raisinganian, 2008): A map $i: X \to Y$ is an isometric, if X and Y are isometry space.

Definition 2.21(Roydon, 2000): The Collection of Borel sets is the smallest $\sigma - algebra$ which contains all of the open sets.

Definition 2.22(Fremlin, 2000): A set which is countable union of closed set is called F_σ (F for closed , σ for sum).

Definition 2.23(Roydon, H.L. 2000): Suppose X is a metric space and $x \in X$ is fixes point if $f(x) = x$.

Definition 2.24(Roydon, H.L. 2000): A net is a mapping of a directed system into a topological space.

Definition 2.25(Paige, D. 2022): A topological space X is Hausdorff if for $A, B \in X, A \cap B = \emptyset$.

MATERIALS AND METHODS

By (Ahmadu & Monsuru, 2020), (Sergio *et.al.,* 2022) and (Jingling *et.al.,* 2021); if X is a topological space, the following represents the collection of $\mathcal{A} - subset$ of X, collection of $\mathcal{A} - closed$ subset of X, collection of closed subsets of X, hyperspace of sub continua of a continuum X and hyperspace of finite unions of convergent sequences in a Hausdorff space X respectively.

1. $\mathcal{A}(X)$ represents the collection of all $\mathcal{A} - subset$ of X.
2. $\mathcal{A}_c(X)$ represents the collection of all $\mathcal{A} - closed$ subset of X.

3. $CL(X)$ represents the collection of closed subsets of X
4. $C(X)$ is called the hyperspace of sub continua of a continuum X.
5. $S(X)$ is called the hyperspace of finite unions of convergent sequences in a Hausdorff space X

By (Victor, 2021), all these are being derived from fundamental notion of collection of subsets of metric space, which is usually designated as $CL(X)$, which reads collection of closed subset of metric or topological space X, which is known as hyperspace in a general sense. This hyperspace can be defined as follows;

Note that according to (Ahmadu & Monsuru, 2020), hyperspace of metric space X can be generated in the following ways:

If $CL(X_1)$ is the collection of closed nonempty subsets of X_1, for each $x_1 \in X_1$ and $X_2 \subset X_1$, $d(x_1, Y) = \inf\{d(x_1, x_2) : x_2 \in X_2\}$. Note that for X_1 and X_2 in hyperspace $CL(X_1)$ and any point $x_1 \in X_1$, there is a point $y \in X_2$ such that $d(x_1, x_2) \leq d_H(x_1, x_2)$, this claim is true due to the following facts (i) point $x_1 \in Y$ only if $x_1 = x_2$ and $d(x_1, x_2) = 0 \leq d_H(x_1, x_2)$ or (ii) $\{x\}$ and X_2 are compact sets that are disjoint in X_1, and there is a point $x_2 \in X_2$ such that $d(x_1, x_2) = d(x_1, x_2) \Longrightarrow d_H(x_1, x_2) = d_H(x_1, x_2)$. Then, $d_H(X_1, X_2) \geq S(X_1, X_2) \geq d_H(x, X_2) = d(x_1, x_2)$.

So, we make use of the notion of collection of closed subsets of metric spaces we defined above with the help of (Ahmad & Takashi, 2018), (Beshimou & Savarova, 2019), (Monsuru *et.al.*, 2020), (Fell, 1962), (Di caprio *et.al.*, 2000), (Murray University, 2021), (Mehdi & Hossein ,2012), (Jerolina *et.al.*, 2022), (Marc, 2009), (Richmond *et.al.*, 2022) and (Lee, 2018) to establish relatively compactness theorem between the hyperspaces.

RESULTS

Here, we prove relatively compactness theorem about metric spaces and some classes of Riemannian manifolds as its application.

Lemma 4.1(Dong, M. 2018)**:** For a compact metric space X and for each $\epsilon > 0$, $Cov(X, 2\epsilon) \leq Cap(X, \epsilon)$.

Now, we are going to use Lemma 4.1 above to prove the Theorem 4.1 below.

Theorem 4.1: Suppose X and Y are compact spaces and CL(X), CL(Y) are hyperspaces of X and Y respectively. If $d_{GH}(\mathrm{CL(X)}, \mathrm{CL(Y)}) < \delta$, then for any $\epsilon > 0$, one has the following estimates:

(i) $Cov(\mathrm{CL(X)}, \epsilon) \geq cov(\mathrm{CL(Y)}, \epsilon + 2\delta)$, and

(ii) $Cap(\mathrm{CL(X)}, \epsilon) \geq Cap(\mathrm{CL(Y)}, \epsilon + 2\delta)$.

Proof: (i) Suppose the hyperspace CL(X) is covered by N $\epsilon -$ balls , say $\{B_i = B(x_i, \epsilon)\}_{i=1}^{N}$ so that N =Cov$(\mathrm{CL(X)}, \epsilon)$**.** By assumption, we have a $\delta -$ Hausdorff approximation $f: \mathrm{CL(X)} \rightarrow \mathrm{CL(Y)}$. Then the balls $\{B(f(x_i), \epsilon + 2\delta)\}$ of radius $\epsilon + 2\delta$ about $f(x_i)$ covers $\mathrm{CL(Y)}$. This prove that $Cov(\mathrm{CL(X)}, \epsilon) \geq cov(\mathrm{CL(Y)}, \epsilon + 2\delta)$.

(ii) To provide a proof for the second inequality, we have to adopt that the hyperspace CL(X) has maximal number of disjoint N $\epsilon -$ balls , say $\{B_i = B(x_i, \epsilon)\}_{i=1}^{N}$ so that N =Cap$(\mathrm{CL(X)}, \epsilon)$**,** that mean there must be $\delta -$ Hausdorff approximation $f: \mathrm{CL(X)} \rightarrow \mathrm{CL(Y)}$ such that the balls $\{B(f(x_i), \epsilon + 2\delta)\}$ of radius $\epsilon + 2\delta$ about $f(x_i)$ has maximal number of disjoint N $\epsilon -$ balls in $\mathrm{CL(Y)}$. This proves that

$$Cap(\mathrm{CL(X)}, \epsilon) \geq Cap(\mathrm{CL(Y)}, \epsilon + 2\delta) \blacksquare$$

Let $(\mathrm{CL(X)}, d_{GH})$ be the set of hyperspaces of every isometric classes of compact metric spaces by means of Gromov – Hausdorff distance. One of the most important properties of Gromov – Hausdorff topology is convergence of a sequence in metric spaces or

Riemannian manifold in hyperspace CL(X). In the next result, we shall use Cauchy criterion to show that the hyperspace CL(X) is complete through the Gromov – Hausdorff distance d_{GH}.

Lemma 4.2: Let X_1 be a metric space and X_2 be nonempty subspace of X_1. Then, the following are satisfy;

(i) X_2 is relatively compact

(ii) In every sequence in X_2 there is a subsequence which converges in X_1

(iii) X_2 is totally bounded and every Cauchy sequence in X_2 converges in X_1.

Lemma 4.3: If X_1 is a nonempty set and $T: X_1 \rightarrow 2^X$ is a map that assign to each element $x \in X_1$ a nonempty subset $\mathcal{A}(x) \in X_1$, then there is a sequence $x_n \in X_1$ such that $x_{n+1} \in \mathcal{A}(x), \forall n \in \mathbb{N}$.

Theorem 4.2: If the hyperspace $(\mathrm{CL(X)}, d_{GH})$ is complete, then it is relatively compact.

Proof: We begin this proof by firstly prove for the completeness of the space $(\mathrm{CL(X)}, d_{GH})$ by Letting $\{X_n\}$ be a Cauchy sequence. It suits to prove that some subsequence of the Cauchy sequence $\{X_n\}$ converges, so that we can assume without loss of detail that $d_{GH}(X_n, X_{n+1}) < 2^{-n}\, for\ all\ n = 1,2, \ldots$. Then, choose metrics $d^{n,n+1}$ on $X_n \cup X_{n+1}$ such that $d_H^{n,n+1}(X_n, X_{n+1}) < 2^{-n}$. With these choice, we can construct metrics $d^{n,m}$ on $X_n \cup X_m$,where $n < m$ as follow;

$$d^{n,m}(x,y) = inf\left\{\sum_{i=1}^{m-1} d^{i,i+1}(x_i, x_{i+1}) : x_i \in X_i\ and\ x_n = x, x_m = y\right\}.$$

This metrics clearly satisfy

$$d^{n,i}(x_n, x_i) \leq d^{n,m}(x_n, x_m) + d^{m,i}(x_m, x_i)$$

Suppose $n \leq m \leq i\ and\ x_n \in X_n, x_m \in X_{m,} x_i \in X_{i.}$ Therefore,

$d_H^{n,m}(X_n, X_m) \leq \sum_{i=1}^{m-1} d_H^{i,i+1}(X_i, X_{i+1}) \leq 2^{-n+1}\ \ if\ n \leq m.$

Let $\hat{X} = \{(x_m): x_m \in X_m and\ d^{n,m}(x_n, x_m) \to 0\ as\ n, m \to 0\}$. So, there is a psuedometric on $\hat{X}$ defined by $d(x_m, y_m) = \lim_{m\to\infty} d(x_m, y_m)$. We contend that metric spaces X, obtained from $\hat{X}$ by identifying points which have zero distance, is the limit of $\{X_n\}$.

Construct a metric d^n on $X_n \cup X$ by $d^n(y, x_m) = \lim_{j\to\infty} sup\, d^{n,m}(y, x_m)$, where $y \in X_n$ and (x_n) represents an elements in X. This is easily seen to give a well-defined metric on $X_n \cup X$. We claim that $d^n_{GH}(X, X_n) \leq 2^{-n+2}$. Let (x_m) represent an element of X. Choose $r \geq n$ such that $d^r(x_r, x_m) < 2^{-n}$, and $y \in X_n$ with $d^r(y, x_r) \leq 2^{-n+2}$.

Thus,

$d^n(y, x_m) = \lim_{j\to\infty} sup d^{n,m}(y, x_m) \leq \lim_{m\to\infty} sup d^{n,r}(y, x_n) +$

$d^{r,m}(x_n, x_m) \leq d^{n,r}(y, x_n) + d^r(x_r, x_m) \leq 2^{-n} + 2^{-n+1} \leq 2^{-n+2}$.

Conversely, suppose $y \in X_n$. We can successifully find $x_m \in X_m, m \geq n$ and $y = x_n$ and $d^{m,m+1}(x_{m,}x_{m+1}) < 2^{-m}$. Then, the sequence (x_m) defined an element in X and by construction

$d^n(y, x_n) = \lim_{m\to\infty} sup d^{n,m}(y, x_m) \leq \lim_{j\to\infty} \sum_{i=n}^{m-1} 2^{-i} = 2^{-n+1}$.

Assume that X_1 and X_2 are nonempty subsets of R^n and $\mathrm{CL}(X_1, X_2)$ be the collection of compact subsets of X_1 and X_2. Let (x_n) be a dense sequence in X_1 and (y_n) be a sequence in $S \subset \mathrm{CL}(X_1, X_2)$. We are going to provides proof of this in three steps as follows, that the sequence (y_n) has a convergent sequence.

Step I : There is a subsequence (p_n) of (y_n) such that the sequence $((p_n(y_n))$ converges in X_2 for all $n \in N$. Provided that $S((x_n))$ is relatively compact, it follow from Lemma 4.3 above that there is a sequence of subsequence (y_{n_t}, i) and $(y_{n_{,t+1}}, i)$ is a subsequence of

(y_{n_t}, i) and the sequence $(y_{n,t,1}, (x_n))$ converges in X_2. Thus, the diagonal subsequence $(p_i) = (y_{n_t}, i)$, this satisfy the requirement of Step I.

Step II: Let (p_i) be as we defined it in Step I above. Then, (p_i) is a Cauchy sequence in $\mathrm{CL}(X_1, X_2)$. Then, by equicontinuity, there is a constant $\delta > 0$ such that for all $g \in \mathcal{A}$ and all $x_{1,}x_2 \in X_1$, we have $d_{X_1}(x_{1,}x_2) < \delta \Longrightarrow d_{X_2}(g(x_1), g(x_2)) < \frac{\epsilon}{3}$. Since the open ball $B_\delta((x_n)$ form an open cover of X_1, there is $k \in \mathbb{N}$ such that $X_1 = \cup_{n=1}^{k} B_\delta((x_n)$. Provided that $(p_i(x_n))$ is Cauchy sequence for each $n \in \mathbb{N}$, there is $K \in \mathbb{N}$ such that for all $i, j, l \in \mathbb{N}$, we get $1 \leq l \leq k, i, j \geq K \Longrightarrow d_{X_2}(p_i(x_n), p_j(x_n)) < \frac{\epsilon}{3}$

...........................(1)

Then, we shall prove that $d(p_i, p_j) < \epsilon$ for all $i, j \geq \mathbb{N}$ by letting $x_{1,} \in X_1$ such that $d_{X_1}(x_{1,}x_n) < \delta$ for the existence of index $k \in \{1,2 \ldots., k\}$. This implies from equation (1) that $d_{X_2}(p_i(x_1), p_j(x_n)) < \frac{\epsilon}{3}, \forall i \in \mathbb{N}$. ....(2)

So, from equation (1) and (2) above, we get $d_{X_2}(p_i(x_1), p_j(x_1)) \leq$

$d_{X_2}(p_i(x_1), p_j(x_n)) + d_{X_2}(p_i(x_n), p_j(x_n)) + d_{X_2}(p_j(x_n), p_j(x_1)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$,

$\forall i, j \geq K$..................(3)

From equation (1) above, we have $d(p_i, p_j) = max d_{X_2}(p_i(x_1), p_j(x_1)) < \epsilon, \forall i, j \geq K$, showing proof of Step II.

Step III: The subsequence (p_i) in step 1 converges in $\mathrm{CL}(X_1, X_2)$. Let $x_1 \in X_1$, from step II , $(p_i(x_1))$ is a Cauchy sequence in hyperspace $\mathcal{A}(x)$. Provided that $\mathcal{A}(x)$ is relatively compact subset of X_2, then the sequence $(p_i(x))$ has a convergent subsequence and hence converges in X_2.

Let $\lim_{i \to \infty} p_i(x) = p(x)$ so that the sequence $p_i(x)$ converges uniformly to $p(x)$ as we can see in step II above, so , $p \in \mathrm{CL}(X_1, X_2)$. This shows that every sequence in $\mathcal{A}$ has a subsequence that converges to $x \in \mathrm{CL}(X_1, X_2)$. This shows that the set $\mathcal{A}$ is precompact by Lemma 4.2 above.

By (Olaf, M, 2021), note that a subspace of a metric space which is relatively compact is totally bounded.

Through this notion, one has the following results;

Theorem 4.3: Suppose $C(X)$ is a closed and bounded subset of metric space X. Then, $C(X)$ is totally bounded if and only if for any $\epsilon > 0$, there exists a finite $\epsilon - net$ in $C(X)$.

Proof: Suppose X is isometry compact metric space and the Gromov – Hausdorff distance d_{GH} becomes a metric on it.

If $C(X)$ is totally bounded. Then, for any $\epsilon > 0$, there is a finite number N_ϵ of $\epsilon - balls$ $\{B_n = B_{GH}(X_n, \epsilon)\}_{n=1}^{N_\epsilon}$ which cover $C(X)$, where $X_n \in C(X)$ and $B_{GH}(X_n, \epsilon) = \{Y \in C: d_{GH}(X_n, Y) < \epsilon\}$. Thus, $\{X_n\}_{n=1}^{N}$ is an $\epsilon - net$ in $C(X)$.

Conversely, for any given $\epsilon > 0$. There exist a finite $\frac{\epsilon}{2} - net$ $\{X_n\}_{n=1}^{N}$ such that for a finite number N_ϵ of $\epsilon - balls, B_{GH}(X_n, \epsilon) = \{Y \in C: d_{GH}(X_n, Y) < \epsilon\}$. Thus, $\{B_n = B_{GH}(X_n, \epsilon)\}_{n=1}^{N_\epsilon}$ cover $C(X)$■

Theorem 4.4 (Gromov relatively compactness theorem): Let $C(X)$ be hyperspace of family of isometry class of compact metric space X; Then, the three statements below are satisfy:

(a) $C(X)$ is relatively compact.

(b) There is a function $f: (0,1] \to (0, \infty)$ such that $Cap(X, \epsilon) \leq N_\epsilon$ for any $\epsilon \in (0,1], X \in C(X)$.

(c) There is a function $f\colon (0,\frac{1}{2}] \rightarrow (0,\infty)$ such that $Cov(X,\epsilon) \leq N_{\epsilon}$, for any $\epsilon \in (0,\frac{1}{2}], X \in C(X)$.

Proof: a ⟹b. Suppose the hyperspace $C(X)$ is relatively compact space. Since relatively compactness implies totally boundedness, then, the hyperspace $C(X)$ is totally bounded. So, for any $\epsilon > 0$, there exists finite set $X_1, X_2, \dots, X_{n(\epsilon)} \in C(X)$ such that for any $X \in C(X)$, there is i such that $d_{GH}(X, X_n) \leq \frac{\epsilon}{4}$, i.e. $\{X_n\}_{n=1}^{n(\epsilon)}$ is a $\frac{\epsilon}{4} - net$ in $C(X)$. Therefore, by **theorem 4.1**, one has $Cap(X,\epsilon) \leq Cap\left(X_n, \epsilon - \frac{2\epsilon}{4}\right) = Cap(X_n, \frac{\epsilon}{2})$. So, just defining $N_{\epsilon} = maxCap(X_n, \frac{\epsilon}{2})$, we get 2.

b⟹ c. This is direct as $Cov(X, 2\epsilon) \leq Cap(X,\epsilon) \leq \ N_{\epsilon}$.

c⟹ a. It will be sufficient to show that for every sequence $\{X_n\}$ in $C(X)$ and every $\epsilon > 0$, there is a sequence $\{X_q^1\}$ where $d_{GH}(X_q, X_n) < \epsilon$ for all elements in $\{X_q^1\}$. Every X_n is covered by at most $N_{\epsilon}\ \epsilon - balls$. Thus, for fixed $N \leq N_{\epsilon}$, there is a subsequence $\{X_q^1\}$ of sequence $\{X_n\}$ such that $\{X_q^1\}$ is covered by exactly $N_{\epsilon} - balls$. Let $\{x_q^l\}_{l=1}^{N}$ be the center of these balls covering $\{X_q^1\}$. For each m, consider the matrix of numbers $\{d(x_q^l, x_q^t)\}_{l,t=1}^{N}$. All these number are bounded by $D\left(X_q^1\right) \leq \epsilon . N$.

According to Pigeon hole principle, there is a subsequence $X_p^{11}\ of\ subsequence\ X_q^1$ such that $|\mathrm{d}(x_p^l, x_p^t) - \mathrm{d}|(x_q^l, x_q^t)| < \frac{\epsilon}{2}$, for l, m , by **lemma 1.1.9**, we have $d_{GH}(X_p^{11}, X_q^{11}\ ) \leq \epsilon$ for all p, q ∎

Theorem 4.5: Given that D is a diameter of relatively compact hyperspace $C(X)$ such that the map $D\colon C(X) \rightarrow \mathbb{R}$ is continuous linear functional under Gromov – Hausdorff topology. Then, (i) $|\mathrm{D}(X_1) - \mathrm{D}(X_2)| \leq d_{GH}(X_1, X_2)$ for any compact metric spaces X_1 and X_2 (ii) $D\colon C(X) \rightarrow \mathbb{R}$ is equip - continuous.

Proof: (i) Suppose $d_{GH}(X_1, X_2) \leq \epsilon$, then by the definition of metric d on $X_1 \cup X_2$ which extends the metrics on X_1 and X_2 such that the Gromov – Hausdorff distance between X_1 and X_2 in $X_1 \cup X_2 \leq 3\epsilon$. Then, for $x_1, x_2 \in X_1$, there exists $x_3, x_4 \in X_2$ such that $d(x_i, x_j) \leq 3\epsilon$. Hence, $d(x_3, x_4) \leq 6\epsilon + d(x_1, x_2)$ by follow triangle inequality. Then by symmetry property, we have $\mathrm{D}(X_2) \leq 6\epsilon + D(X_1) \Longrightarrow |\mathrm{D}(X_1) - \mathrm{D}(X_2)| \leq d_{GH}(X_1, X_2)$

(ii). Suppose $A \in D$ and $x_1, x_2 \in X_1$ such that $d_{GH}(x_1, x_2) < \delta$. Since from the definition of relatively compactness, it follows that there is an index $n \in \{1,2, \dots k\}$ such that $d(A, A_n) < \frac{\epsilon}{3}$. Therefore, $d_{X_2}(A(x), A_n(x)) < \frac{\epsilon}{3}$ and $d_{X_2}(A(x_1), A_n(x_1)) < \frac{\epsilon}{3}$. Furthermore, this follow from the uniformly continuity that $d_{X_2}(A_n(x), A_n(x_1)) < \frac{\epsilon}{3}$. Then through triangle inequality, we get $d_{X_2}(A(x), A_n(x)) \leq d_{X_2}\big(A(x), A_n(x)\big) + d_{X_2}(A_n(x), A_n(x_1)) + d_{X_2}(A(x_1), A_n(x_1)) \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$. This prove that the map D is equi- continuous.

CONCLUSION

This paper defined some geometric properties in conjunction with some selected topological properties such as totally boundedness, compactness, relatively compactness and completeness in Gromov sense on a Riemannian Manifold.

REFERENCES

Ahmadu, K., and Monsuru, A. M. (2021). On some topological properties on a certain hyperspace topology. *IOSR Journal of Mathematics*. 1*7(1)*, 34-38.

Alexander, V. O., and Oscag., S. (2017). Variation of selective separability and tightness in function spaces with set open topology. *Topology and its applications 217. 38 – 50.*

Ahmad, O., and Takashi, N. (2018). On z- compact spaces and some functions. BOI. Soc. Paran, Math, .36(5). 121 – 130.

Beshimou, R.B., and Savarova, D.T. (2019). Topological Properties of hyperspaces. *Bullettin of National University of Uzbekistan: Mathematics and Natural Sciences. Vol. 2*

Dong, M. (2018). Group Actions from measure theoretical viewpoint. *Ph.D. thesis, Chungnan National University in Daejeon.*

Di Caprio, D., and Meccariello, E. (2000). Notes on separation axioms in hyperspaces. *I.Q and A in general topology. Vol. 18. 65 – 86.*

Dong, C., and Gabjin, Y. (2000). "Gromov – Hausdorff Topology and its application to Riemannia Manifold.

Engelking, R., and Siekluchi, K. (1992). Topology: A geometric approach. *Heldermann Verlag.*

Fremlin, D.H. (2019). Measure theory. *Volume 5. University of Essex.*

Facundo, M and Zhengchao (2021). Charaterization of Gromov – type geodesics. *Department of Mathematics and Computer Science and Engineering, The Ohio State University.*

Facundo, M. (2012). Some properties of Gromov – Hausdorff distance. *Discrete compute Geom. 48. 416 – 440.*

Fernandez, U.M. (2018). The segre cone of Banach spaces and multilinear mappings. *Linear multilinear Algebra. 10.1080/03081087.2018.1509938.*

Fell, J. (1962). A Hausdorff topology for the closed subset of a locally compact and non – Hausdorff spaces. *Proc. Amer.Math.Soc.13, 472 – 476.*

Jerolina, F., Neeraj, M., Anas, S., Marija, P., and Zoran, D. (2022). The extended cone b – metric – like spaces over Banach algebra and some application. *MDPI Journal. Vol 10.*

Lee, G.T. (2018). Abstract algebra: An introductory course. *Springer Undergraduate Mathematics Series, Canada. Doi: 10.1007/978 – 3 – 319.*

Masaki, M. (2022) "Mean Hausdorff Dimension of some infinite dimensional fractals"2020 Mathematics subject classification.

Monsuru, A.M., Ahmadu, K., and Balla, M. Y. (2020). On the comparative study of compactness and some of its relative notions in Metric and Topological spaces. *International Journal of scientific and Research Publication, Vol. 10 Issue 10. 659 – 665.*

Murray. (2021). The 54th Spring Topology and Dynamic Conference. *Murray State University. USA.*

Marc, C. (2009). Volume and Topology

Mehdi, A., and Hossein, S. (2012). Examples in Cone Metric spaces: *A survey. Middle – East Journal of Scientific Research. 11(12): 1636 – 1640.*

Olaf, M. (2021) "Gheeger – Gromov compactness theorem for Manifold with boundary. Adv. Math. 224 – 240.

Olaf, M. (2021). Gheeger – Gromov compactness for manifold with boundary. *Adv. Math. 224 – 240.*

Oxtoby, J.C. (1971). Measure and category: A survey of analogies between topological and measure spaces. *Springer.*

Olga, G.M., and Peter W.M. (1991). The Riemannian manifold of all Riemannia metrics. *Quarterly Oxford Journal of Mathematics 42(183 - 202).*

Paige, D. (2022). Hausdorff Dimension and Projection

Pratulananda, D., Sadip, P., and Nayan, A. (2021). On certain notions of precompactness, continuity, and Lipschitz functions. *Cornell University.*

Perelman, G. (1995). Spaces with curvature bounded below. *Proceedings of international congress of Mathematics.*

Roydon, H.L. (2000). Real Analysis. *New Delhi.*

Richmond, K.G., William, O. D., and Fred, A. (2022). Topological spaces with emphases on the sphere and its application. *Full Length Research Article.*

Knox, K.S. (2013) "Compactness theorems for Riemannian Manifolds with Boundary and Application" Ph.D. thesis, Stony brook University.

Sergio, A. (2002) "A compactness theorem for scaler – flat metrics on manifold with boundary" Cal. Var. Partial differential equations 41(2011), 341 – 386.

Satvik, S. (2022) "A brief discourse on Hausdorff dimension and self – similarity" The American Mathematical Monthly. Vol.129, Issue 9, 123 – 144.

Victor, D., Natalia, J., and Ananda, L. (2021). Some Notes On Induced Functions and Group Actions On Hyperspace. *2010 Mathematics Subjects Classification. Department De Mathematicas, Facultad de Ciencias, Mexico. Volume 4, page 1-23.*

Valov, V. (2020). Homogeneous Metric ANR Compacta. *2000 Mathematical Classification.*

Willard, S. (2004). General topology. *Additson – Wesley.*

Wallace, A. H. (2007). Algebraic topology: Homology and cohomology. *Benjamin, 1972, Dover.*

Watson, W.S. (1981). Pseudo compact metacompact spaces are compact. *Proc. Amer. Math. Soc,81, 151 - 152*