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GENERALIZATION AND EXTENSION OF LIU,H. AND XU,S. AND, JIANDONG WANG RESULTS IN CONE METRIC SPACES OVER BANACH ALGEBRAS FOR GENERALIZED LIPSCHITZ CONDITIONS

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Abstract

This article aims to present and establish some fixed-point results for a mapping satisfying generalized Lipschitz conditions and appealing to the normality of the cone. Our results extend and generalize several known results from the existing literature.

Keywords: Cone Metric Space Over Banach Algebra, Generalized Lipschitz Mappings, Fixed Point

INTRODUCTION

The Banach contraction theory was established in 1922 by Banach S. This theory is also referred to as the Banach fixed point theorem. Most recentlyThe idea of cone metric spaces, which is an extension of metric spaces, was introduced by Huang and Zhang in 2007. In this concept, the real numbers are replaced by a Banach space that is ordered.

They proved some fixed point theorems of contractive mappings on cone metric spaces under the assumption that cones are normal. The findings of Huang and Zhang's research in 2007 were expanded upon by Rezapour and Halmbarani in 2008 by removing the assumption of normality in the cone. As a result, several researchers have extended the outcomes of Huang and Zhang's study and have looked into fixed and common fixed point theorems for both normal and non-normal cones.

A few years ago, Liu and Xu (2013) proved some fixed point theorems for generalized Lipschitz maps in cone metric spaces, under weaker and more natural conditions.

In 2014, Shaoyuan Xu and S. Radenovic improved and generalized the results of Liu,H. and Xu,S.(2013),and obtained some fixed point results for generalized Lipchitz mapping in cone metric spaces over Banach algebras without the assumption of normality.

Huaping, H.and S. Radenovic (2015) obtained some common fixed point theorems in same space. In 2016, Qi Yan et al. , obtained some fixed point and common fixed point results of comparable maps satisfying certain contractive conditions on partially ordered cone metric spaces over Banach algebras.

In 2018, Cho S-H established a new fixed point theorem and presented an example. In the same year, Yan H. Shaoyuan and Xu obtained fixed point and common fixed point results for generalized Lipschitz conditions on c-distance without relying on continuity. Huaping H. et al. conducted research in 2015 on c-distance for generalized Lipschitz conditions on cone metric spaces with Banach algebras and proved a common fixed point. In 2017, Ahmad A. et al. improved an analog of Banach and Kannan fixed point theorems by extending the Lipschitz constant in generalized Lipschitz mapping Banach algebra. Their work was a response to the open problems proposed by Sastry et al. (2012) in cone metric spaces.

The purpose of this manuscript, we establish and generalize, extend and improve some important results in the literature of Liu, H. (2013), Xu, S. (2016), and Wang, J. (2016).

PRELIMINARY NOTES

We will review some basic concepts and definitions from Huang and Zhang (2007), and Liu,H. and Xu,S.(2013), as follows:

Let A be a real Banach algebra an operation of multiplication is defined sbject to the following properties(for all $x, y, z \in A, \alpha \in R$):

(1) $(xy)z = x(yz)$;

(2) $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;

(3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$;

(4) $\|xy\| \leq \|x\|\|y\|$.

An algebra A with unit element e, unital algebra, i.e. multiplicative identity e such that $ex = xe = x$ for all $x \in A$. An element $x \in A$ is said to be invertible if there is an inverse element $y \in A$ such that $xy = yx = e$. the inverse of x is denoted by x^1.

The following proposition is well-known (see[Liu,H. and Xu, S.(2013),and Rudin W. .(1991),]).

Proposition 1: Let A be a real Banach algebra with unit e and $x \in A$. If the spectral radius $r(x)$ of x is less than1, that is,

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|x^n\|^{1/n} < 1, \text{ then}$$

$e - x$ is invertible. $e.t.$

$$(e - x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

If $r(x) < 1$, then $\|x^n\| \to 0 (n \to \infty)$. Because $r(x)$ denotes the spectral radius of $x \in A$. Now, let us recall of cone and partial ordering for a Banach algebra A. A subset P of A is called a cone of A if,

(i) P is non empty closed and $\{0, e\} \subset$ P,where 0 denotes the zero element of A;

(ii) $\alpha P + \beta P \in P$ for all non negative real numbers α, β;

(iii) $P^2 = PP \subset P$;

(iv) $P \cap (-P) = \{\theta\}$.

Given a cone P ⊂ E, we define a partial ordering ≤ on E with respect to P by $x \leq y$ if and only if $y - x \in P$.We shall write $x \ll y$ if $y - x \in intP$(where int P denotes the interior of P). If $intP \neq \emptyset$, then cone P is solid.

The cone P called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$\theta \leq x \leq y \Rightarrow \| x \| \leq K \| y \|.$$

The least positive number satisfying the above is called the normal constant of P.

Definition: 2 (see [Huang and Zhang (2007) and Liu,H. and Xu, S.(2013)]): Let X be a non-empty set. Let A be Banach algebra and $P \subset$ A be a cone. Suppose the mapping $d: X \times X \longrightarrow A$ satisfies

(i) $0 < d(x, y)$for all $x, y \in X$ and $(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$for all $x, y \in X$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)$for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space over a Banach algebra .

We present some examples in the following:

Example 3: Let $A = \{a = \left(a_{ij}\right)_{33}: a_{ij} \in \mathbb{R}, 1 \leq i, j \leq 3\}$ and

$$\|a\| = \frac{1}{3} \sum_{1 \leq j, j \leq 3} \left|a_{ij}\right|.$$

Take a cone $P = \{a \in A: a_{ij} \geq 0, 1 \leq i, j \leq 3\}$ in A.Let $X = (1,2,3\}$. Define a mapping $d: X \times X \longrightarrow A$ by $d(1,1) = d(3,3) = (0)_{33}$ and

$$d(1,2) = d(2,1) = \begin{pmatrix} 2\ 2\ 4 \\ 3\ 4\ 3 \\ 2\ 5\ 3 \end{pmatrix},$$

$$d(3,1) = d(1,3) = \begin{pmatrix} 5\ 4\ 6 \\ 6\ 7\ 5 \\ 5\ 6\ 4 \end{pmatrix},$$

$$d(2,3) = d(3,2) = \begin{pmatrix} 4\ ,5\ ,6 \\ 5\ ,7\ ,6 \\ 7\ ,7\ ,5 \end{pmatrix}.$$

Then (X, d) is a cone metric space over Banach algebra.

Definition: 4 [Huang and Zhang (2007) and Liu,H. and Xu, S.(2013),]: Let (X, d) be a cone metric space over a Banach algebra A ,$x \in X$ and $\{x_n\}_{n \geq 1}$ be a sequence in X. Then

(1) $\{x_n\}_{n \geq 1}$ Converges to x whenever for every$c \in E$ with $\theta \ll c$,there is a natural number N such that$d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $lim_{n \to \infty} x_n \quad x$ or $x_n \to x, (n \to \infty)$

(2) $\{x_n\}_{n \geq 1}$ is said to be a Cauchy sequence if for every $c \in A$ with θ<< c,if there is a natural number N such that for all $n, m > N, d\left(x_{n,} x_m\right) \ll c$;

(3) (X, d) is said to be a complete cone metric space over Banach algebra if every Cauchy sequence in X is convergent in X.

Lemma 5(see [Rudin,W.(1991),]): Let A be Banach algebra with a unit e.If $x, y \in A$ and x commute with y, then

$$r(x+y) \leq r(x)+r(y) \text{ and}$$

$$r(x\, y) \leq r(x)\, r(y).$$

Lemma 6 (see [Rudin W.(1991)]): Let A be a Banach algebra with a unit e and k be a vector in A. If $0 \leq r(k) < 1$, then we have

$$r((e-k\,)^{-1}) \leq (1-r(k))^{-1}.$$

Definition 7: Let (X, d) be a cone metric space over a Banach algebra A and P be a normal cone with a normal constant K. Then a mapping $S: X \rightarrow X$ is called generalized Lipschitz mapping if there exists a vector $k \in P$ with $k < e$ and for all $x, y \in X$,we have

$$d(Sx, Sy) \leq kd(x, y)$$

Lemma 8: Suppose P is normal cone with constant K in (X, d) with Banach algebra A. If $x \leq y$, then $r(x) \leq r(y)$.

Proof: Let (X, d) be a cone metric space over a Banach algebra A and let P be a normal cone with a normal constant K such that $x \leq y$ means $x^n \leq y^n$, then we have prove to prove that $r(x) \leq r(y)$. Since $x \leq y$ means $x^n \leq y^n$. So, by definition of normality

$$\|x^n\| \leq K\|y^n\|. \text{ This implies} \|x^n\|^{1/n} \leq K^{1/n} \left\|y^{1/n}\right\|^{1/n}.$$

Thus

$$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n} \leq \lim_{n\to\infty} (K^{1/n} \left\|y^{1/n}\right\|^{1/n})$$

$$= \lim_{n\to\infty} \left\|y^{1/n}\right\|^{1/n} = r(y).$$

Lemma 9: Let (X, d) be a cone metric space over a Banach algebra A and let P be a normal cone with a normal constant K. Let $x, k \in P$ hold $x \leq kx$. If $r(k) < 1$, then $x = 0$.

Proof: Since$r(k) < 1$. So, $r(k) = \lim_{n\to\infty} \|k^n\|^{1/n} < 1$, then there exists $\alpha > 0$, such that $\lim_{n\to\infty} \|k^n\|^{1/n} < \alpha < 1$. letting n be big enough, we get $\|k^n\|^{\frac{1}{n}} \leq \alpha$,i.e. $\|k^n\| \leq \alpha^n \rightarrow 0 (n \rightarrow \infty)$. Thus $\|k^n\| \rightarrow 0 (n \rightarrow \infty)$. Note that by definition of normality of P and $x \leq kx \leq k^2x \leq \cdots \ldots \ldots \leq k^n x$, it follows that

$$\|x\| \leq K\|k^n\|\,\|x\| \rightarrow 0 (n \rightarrow \infty), \text{ This implies } x = 0$$

RESULTS

Theorem 10: Let (X,d)be a complete cone metric spaces over Banach algebra A, P be a normal cone with a normal constant$K$. Suppose the mapping $S: X \to X$ satisfies the generalized Lipschitze conditions

$$\boldsymbol{d}(Sx,Sy) \leq \lambda_1 d(x,y) + \lambda_2 d(Sx,x) + \lambda_3 q(Sy,y) + \lambda_4 d(Sx,y) \ \ldots\ldots\ldots \ (10.1)$$

for all $x,y \in X$, where $\lambda_i \in P(i = 1,2,3, \ldots \ldots ..)$ are generalized Lipschitz constants with $r(\lambda_2 + \lambda_4) + r(\lambda_1 + \lambda_3 + \lambda_4) < 1$. If $\lambda_2 + \lambda_4$ commutes with $\lambda_1 + \lambda_3 + \lambda_4$ then S has a unique fixed point in X. And for any $x_0 \in X$, iterative sequence $\{S^n x\}$ converges to the fixed point.

Proof: Let x_0 is an arbitrary point in X and set $x_n = Sx_{n-1} = S^n x_0, n \geq 1$.

Put $x = x_n$ and $y = x_{n-1}$ from (10.1), we have

$$\begin{aligned} d(x_{n+1},x_n) &\leq d(Sx_{n,}Sx_{n-1}) \\ &\leq \lambda_1 d(x_n,x_{n-1}) + \lambda_2 d(Sx_n,x_n) + \lambda_3 d(Sx_{n-1},x_{n-1}) \\ &\quad + \lambda_4 d(Sx_n,x_{n-1}) \\ &\leq \lambda_1 d(x_n,x_{n-1}) + \lambda_2 d(x_{n+1},x_n) + \lambda_3 d(x_n,x_{n-1}) \\ &\quad + \lambda_4 d(x_{n+1},x_{n-1}) \\ &\leq \lambda_1 d(x_n,x_{n-1}) + \lambda_2 d(x_{n+1},x_n) + \lambda_3 d(x_n,x_{n-1}) \\ &\quad + \lambda_4 [d(x_{n-1},x_n) + d(x_n,x_{n+1})] \\ &\leq (\lambda_1 + \lambda_3 + \lambda_4) d(x_{n-1},x_n) + (\lambda_2 + \lambda_4) d(x_n,x_{n+1}) \end{aligned}$$

This means that

$$(e - \lambda_2 - \lambda_4)\, d(x_{n+1},x_n) \leq (\lambda_1 + \lambda_3 + \lambda_4) d(x_{n-1},x_n) \qquad (10.2)$$

Since $r(\lambda_2 + \lambda_4) \leq r(\lambda_2 + \lambda_4) + r(\lambda_1 + \lambda_3 + \lambda_4) < 1$, then by proposition 1

$(e - \lambda_2 - \lambda_4)$ is invertible. Furthermore,

$$(e - \lambda_2 - \lambda_4)^{-1} = \sum_{i=0}^{\infty} (\lambda_2 + \lambda_4)^i \ \ldots \qquad (10.3)$$

Put $h = (e - \lambda_2 - \lambda_4)^{-1}.(\lambda_1 + \lambda_3 + \lambda_4)$. As $\lambda_2 + \lambda_4$ commutes with $\lambda_1 + \lambda_3 + \lambda_4$, it follows that

$$\begin{aligned} (e - \lambda_2 - \lambda_4)^{-1}.(\lambda_1 + \lambda_3 + \lambda_4) &= \left(\sum_{i=0}^{\infty} (\lambda_2 + \lambda_4)^i\right) (\lambda_1 + \lambda_3 + \lambda_4) \\ &= (\lambda_1 + \lambda_3 + \lambda_4) \left(\sum_{i=0}^{\infty} (\lambda_2 + \lambda_4)^i\right) \\ &= (\lambda_1 + \lambda_3 + \lambda_4)\,(e - \lambda_2 - \lambda_4)^{-1}. \ \ldots\ldots \ (10.4) \end{aligned}$$

That is $(e - \lambda_2 - \lambda_4)^{-1}$ commutes with $(\lambda_1 + \lambda_3 + \lambda_4)$. Then by Lemma 2.6 and Lemma 6, we obtain

$$
\begin{aligned}
r(h) &= r[(e-\lambda_2-\lambda_4)^{-1}.(\lambda_1+\lambda_3+\lambda_4)] \\
&\le r[(e-\lambda_2-\lambda_4)^{-1}].\ r(\lambda_1+\lambda_3+\lambda_4) \\
&\le [1-(\lambda_2+\lambda_4)]^{-1}.r(\lambda_1+\lambda_3+\lambda_4) \\
&= \frac{1}{1-(\lambda_2+\lambda_4)}.r(\lambda_1+\lambda_3+\lambda_4) \\
&< 1
\end{aligned}
$$

Which illustrates that $(e-h)^{-1} = \sum_{i=0}^{\infty}(h)^i$ and $\|h^n\| \to 0(n\to\infty)$. Thus for, we get

$$
\begin{aligned}
d(x_{n+1},x_n) &\le (e-\lambda_2-\lambda_4)^{-1}.(\lambda_1+\lambda_3+\lambda_4)\, d(x_n,x_{n-1}) \\
&= h\, d(x_n,x_{n-1}) \le h^2 d(x_{n-1},x_{n-2}) \le \cdots \ldots . \le h^n d(x_0,x_1)
\end{aligned}
$$

Now, Let $m > n \ge 1$. then it follows that

$$
\begin{aligned}
d(x_{n,}\, x_m) &\le d(x_{n,}x_{n+1}) + d(x_{n+1,}x_{n+2}) + \ldots\ldots\ldots\ldots + d(x_{m-1,}x_m) \\
&\le (h^n + h^{n+1} + \cdots \ldots \ldots \ldots \ldots \ldots + h^{m-1})d(x_{0,}x_1) \\
&\le (e + h + h^{n+1} + \cdots \ldots \ldots \ldots \ldots \ldots + h^{m-n-1})d(x_{0,}x_1) \\
&\le \frac{h^n}{1-h} d(x_{0,}x_1) \\
&\le (e-h)^{-1}h^n\, d(x_{0,}x_1) \qquad (10.5)
\end{aligned}
$$

Since P is normal with normal constant$K$,and note that $\|h^n\| \to 0(n\to\infty)$, we have

$$
\begin{aligned}
\|d(x_n,x_m)\| &\le K[\|(e-h)^{-1}h^n\, d(x_{0,}x_1)\| \\
&\le K[\|(e-h)^{-1}\|\, \|h^n\|\, \|d(x_{0,}x_1)\| \to 0(n\to\infty).
\end{aligned}
$$

Hence $\{x_n\}$ is a Cauchy sequence in$X$. Since X is complete, there exists $u \in X$ such that $x_n \to u(n\to\infty)$. To verify$Su = u$, we have

$$
\begin{aligned}
d(Su,u) &\le d(Su,Sx_n) + d(Sx_n,u) \\
&\le \lambda_1 d(u,x_n,) + \lambda_2 d(Su,u) + \lambda_3 d(Sx_n,x_n) + \lambda_4 d(Su,x_n) + d(x_{n+1,}u) \\
&\le \lambda_1 d(u,x_n,) + \lambda_2 d(Su,u) + \lambda_3 d(x_{n+1},x_n) + \lambda_4 d(Su,x_n) + d(x_{n+1,}u) \\
&\le \lambda_1 d(u,x_n,) + \lambda_2 d(Su,u) + \lambda_3[d(u,x_n) + d(x_{n+1,}u) \\
&\quad + \lambda_4[d(u,x_n) + d(Su,u)] + d(x_{n+1,}u)]
\end{aligned}
$$

Implies that $(e-\lambda_2-\lambda_4)d(Su,u) \le (\lambda_1+\lambda_3+\lambda_4)d(u,x_n) + (1+\lambda_3)d(x_{n+1},u)$

Note that, $(e-\lambda_2-\lambda_4)$ is invertible, then by the normality of, we have

$$
\begin{aligned}
\|d(Su,u)\| &\le K[\{\|e-(\lambda_2+\lambda_4)\|\}^{-1}\|(\lambda_1+\lambda_3+)\|\|d(u,x_n)\| \\
&\quad + \|(e+\lambda_3)\|\|d(x_{n+1},u)\|] \to 0(n\to\infty).
\end{aligned}
$$

Hence, $\|d(Su,u)\| = 0$.This implies $Su = u$. So, u is a fixed point of S in X.

Now, v is another fixed point of S in X. Then

$$d(u,v) \leq d(Su,Sv)$$
$$\leq \lambda_1 d(u,v) + \lambda_2 (Su,u) + \lambda_3 d(Sv,v) + \lambda_4 d(Su,v)$$
$$= (\lambda_1 + \lambda_4)\, d(u,v).$$

Thus $\|d(u,v)\| \leq K\|(\lambda_1 + \lambda_4)\|\,\|d(u,v)\| \to 0(n \to \infty)$.

Then $d(u,v) = 0$, which implies that $u = v$. Therefore, u is an unique fixed point of S in X. This theorem is proved.

Theorem 11: Let (X,d)be a complete cone metric spaces over Banach algebra A, P be a normal cone with a normal constant$K$. Suppose the mapping $S: X \to X$ satisfies the generalized Lipschitze conditions

$$\boldsymbol{d}(Sx,Sy) \leq \lambda_1 d(x,y) + \lambda_2 d(Sx,x) + \lambda_3 d(Sy,x)$$
$$+\lambda_4 d(Sx,y) + \lambda_5 d(Sy,y) \ldots \qquad (11.1)$$

for all $x, y \in X$, where $\lambda_i \in P (i = 1,2,3,4)$ are generalized Lipschitz constants with $r(\lambda_2 + \lambda_4) + r(\lambda_1 + 2\lambda_3 + \lambda_4 + \lambda_5) < 1$. If $\lambda_2 + \lambda_4$ commutes with $\lambda_1 + \lambda_3 + \lambda_4$, then fixed point S is unique in X. And for any $x_0 \in X$, iterative sequence $\{S^n x\}$ converges to the fixed point.

Proof: Let x_0 is an arbitrary point in X and set$x_n = Sx_{n-1} = S^n x_0$, $n \geq 1$.Put $x = x_n$ and $y = x_{n-1}$ from (3.1.1), we have

$$d(x_{n+1}, x_n) \leq d(Sx_n, Sx_{n-1})$$
$$\leq \lambda_1 d(x_n, x_{n-1}) + \lambda_2 d(Sx_n, x_n) + \lambda_3 d(Sx_{n-1}, x_n) + \lambda_4 d(Sx_n, x_{n-1})$$
$$+ \lambda_5 d(Sx_{n-1}, x_{n-1})$$
$$\leq \lambda_1 d(x_n, x_{n-1}) + \lambda_2 d(x_{n+1}, x_n) + \lambda_3 [d(x_n, x_{n-1}) + d(x_{n-1}, x_n)]$$
$$+\lambda_4 [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + \lambda_5 d(x_n, x_{n-1})$$
$$\leq (\lambda_1 + 2\lambda_3 + \lambda_4 + \lambda_5) d(x_{n-1}, x_n) + (\lambda_2 + \lambda_4) d(x_n, x_{n+1})$$

This means that

$$(e - \lambda_2 - \lambda_4)\, d(x_{n+1}, x_n) \leq (\lambda_1 + 2\lambda_3 + \lambda_4 + \lambda_5) d(x_{n-1}, x_n) \ldots\ldots\ldots \quad (11.2)$$

Since $r(\lambda_2 + \lambda_4) \leq r(\lambda_2 + \lambda_4) + r(\lambda_1 + 2\lambda_3 + \lambda_4 + \lambda_5) < 1$, then by proposition 1, $(e - \lambda_2 - \lambda_4)$ is invertible. Furthermore,

$$(e - \lambda_2 - \lambda_4)^{-1} = \sum_{i=0}^{\infty} (\lambda_2 + \lambda_4)^i \ldots \qquad (11.3)$$

Taking $h = (e - \lambda_2 - \lambda_4)^{-1}.(\lambda_1 + 2\lambda_3 + \lambda_4 + \lambda_5)$. As $\lambda_2 + \lambda_4$ commutes with $\lambda_1 + 2\lambda_3 + \lambda_4 + \lambda_5$ it follows that

$$(e - \lambda_2 - \lambda_4)^{-1}.(\lambda_1 + 2\lambda_3 + \lambda_4 + \lambda_5) = \left(\sum_{i=0}^{\infty} (\lambda_2 + \lambda_4)^i\right)(\lambda_1 + 2\lambda_3 + \lambda_4 + \lambda_5)$$

$$= (\lambda_1 + 2\lambda_3 + \lambda_4 + \lambda_5)\left(\sum_{i=0}^{\infty}(\lambda_2 + \lambda_4)^i\right)$$
$$= (\lambda_1 + 2\lambda_3 + \lambda_4 + \lambda_5)\,(e - \lambda_2 - \lambda_4)^{-1}.$$

That is $(e - \lambda_2 - \lambda_4)^{-1}$ commutes with $(\lambda_1 + 2\lambda_3 + \lambda_4 + \lambda_5)$. Then by Lemma 6 and Lemma 7, we obtain

$$\begin{aligned} r(h) &= r[\,(e - \lambda_2 - \lambda_4)^{-1}.(\lambda_1 + 2\lambda_3 + \lambda_4 + \lambda_5)] \\ &\le r[(e - \lambda_2 - \lambda_4)^{-1}].\ r(\lambda_1 + 2\lambda_3 + \lambda_4 + \lambda_5) \\ &\le [1 - (\lambda_2 + \lambda_4)]^{-1}.r(\lambda_1 + 2\lambda_3 + \lambda_4 + \lambda_5) \\ &= \frac{1}{1-(\lambda_2+\lambda_4)}.r(\lambda_1 + 2\lambda_3 + \lambda_4 + \lambda_5) \\ &< 1 \end{aligned}$$

Which illustrates that $(e - h)^{-1} = \sum_{i=0}^{\infty}(h)^i$ and $\|h^n\| \to 0 (n \to \infty)$. Thus for, we get

$$\begin{aligned} d(x_{n+1}, x_n) &\le (e - \lambda_2 - \lambda_4)^{-1}.(\lambda_1 + 2\lambda_3 + \lambda_4 + \lambda_5)\, d(x_n, x_{n-1}) \\ &= h\, d(x_n, x_{n-1}) \le h^2 d(x_{n-1}, x_{n-2}) \le \cdots \ldots . \le h^n d(x_0, x_1) \end{aligned}$$

Now, Let $m > n \ge 1$. then it follows that

$$\begin{aligned} d(x_{n,} x_m) &\le d\left(x_{n,} x_{n+1}\right) + d\left(x_{n+1,} x_{n+2}\right) + \ldots\ldots\ldots.. + d(x_{m-1,} x_m) \\ &\le (h^n + h^{n+1} + \cdots \ldots \ldots \ldots \ldots \ldots + h^{m-1}) d(x_{0,} x_1) \\ &\le (e + h + h^{n+1} + \cdots \ldots \ldots \ldots \ldots \ldots + h^{m-n-1}) d(x_{0,} x_1) \\ &\le \frac{h^n}{1-h}\, d\left(x_{0,} x_1\right) \\ &\le (e - h)^{-1} h^n\, d\left(x_{0,} x_1\right) \qquad (11.4) \end{aligned}$$

Since P is normal with normal constant K, and note that $\|h^n\| \to 0(n \to \infty)$, we have

$$\begin{aligned} \|d(x_n, x_m)\| &\le K[\left\|(e - h)^{-1} h^n\, d\left(x_{0,} x_1\right)\right\|] \\ &\le K[\|(e - h)^{-1}\|\ \|h^n\|\ \left\|d\left(x_{0,} x_1\right)\right\|] \to 0(n \to \infty). \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in$X$. Since X is complete, there exists $u \in X$ such that $x_n \to u(n \to \infty)$. To verify$Su = u$, we have

$$\begin{aligned} d(Su, u) &\le d(Su, Sx_n) + d(Sx_n, u) \\ &\le \lambda_1 d(u, x_n,) + \lambda_2 d(Su, u) + \lambda_3 d(Sx_n, u) + \lambda_4 d(Su, x_n) + \lambda_5 d(Sx_n, x_n) \\ &\quad + d(x_{n+1,} u) \\ &\le \lambda_1 d(u, x_n,) + \lambda_2 d(Su, u) + \lambda_3 d(x_{n+1}, u) + \lambda_4 d(Su, x_n) + d(x_{n+1,} x_n) \\ &\quad + d(x_{n+1,} u) \\ &\le \lambda_1 d(u, x_n,) + \lambda_2 d(Su, u) + \lambda_3 d(x_{n+1}, u) + \lambda_4 [d(u, x_n) + d(Su, u)] \\ &\quad + \lambda_5 [d(u, x_n) + d\left(x_{n+1,} u\right)] \quad + d\left(x_{n+1,} u\right) \end{aligned}$$

Implies that,

$(e-\lambda_2-\lambda_4)d(Su,u) \leq (\lambda_1+\lambda_4+\lambda_5)d(u,x_n) + (1+\lambda_3+\lambda_5)d(x_{n+1},u)$

Note that, $(e-\lambda_2-\lambda_4)$ is invertible, then by the normality of we have

$$\|d(Su,u)\| \leq K[\{\|e-(\lambda_2+\lambda_4)\|\}^{-1}\|(\lambda_1+\lambda_4+\lambda_5)\|\|d(u,x_n)\|$$
$$+\|(e+\lambda_3+\lambda_5)\|\|d(x_{n+1},u)\|] \to 0 (n\to\infty).$$

Hence, $\|d(Su,u)\| = 0$.This implies $Su = u$. So, u is a fixed point of S in X.

Now, v is another fixed point of S in X. Then

$$\begin{aligned} d(u,v) &\leq d(Su,Sv) \\ &\leq \lambda_1 d(u,v) + \lambda_2(Su,u) + \lambda_3 d(Sv,u) + \lambda_4 d(Su,v) + \lambda_5 d(Sv,v) \\ &= (\lambda_1+\lambda_3+\lambda_4)\, d(u,v) \end{aligned}$$

Thus $\|d(u,v)\| \leq K\|(\lambda_1+\lambda_3+\lambda_4)\|\,\|d(u,v)\| \to 0(n\to\infty)$. Then $d(u,v)=0$, which implies that $u=v$. Therefore, u is an unique fixed point of S in X.

Theorem 12: Let (X,d)be a complete cone metric spaces over Banach algebra A, P be a normal cone with a normal constant$K$. Suppose the mapping $S: X\to X$ satisfies the generalized Lipchitz conditions

$$\boldsymbol{d}(Sx,Sy) \leq Aq(x,y) + Bd(Sx,x) + Cd(Sy,y) + D[d(Sx,y) + d(Sy,x)]\ldots \quad (12.1)$$

for all $x,y \in X$, where $A,B,C,D \in P$ are generalized Lipschitz constants with$r(B+D)$ $+r(A+C+D) < 1$. If $r(B+D)$commutes with $A+C+D$, then S has a unique fixed point in X. And for any $x_0 \in X$, iterative sequence $\{S^n x\}$ converges to the fixed point.

Proof: Let x_0 is an arbitrary point in X and set $x_n = Sx_{n-1} = S^n x_0, n\geq 1$.

Put $x = x_n$ and $y = x_{n-1}$ from (12.1), we have

$$\begin{aligned} d(x_{n+1},x_n) &\leq d(Sx_n, Sx_{n-1}) \\ &\leq Ad(x_n,x_{n-1}) + Bd(Sx_n,x_n) + Cd(Sx_{n-1},x_{n-1}) \\ &\quad + D[d(Sx_n,x_{n-1}) + d(Sx_{n-1},x_n)] \\ &\leq Ad(x_n,x_{n-1}) + Bd(x_{n+1},x_n) + Cd(x_n,x_{n-1}) \\ &\quad + D[d(x_{n+1},x_{n-1}) + d(x_n,x_n)] \\ &\leq Ad(x_n,x_{n-1}) + Bd(x_{n+1},x_n) + Cd(x_n,x_{n-1}) \\ &\quad + D[d(x_n,x_{n-1}) + d(x_n,x_{n+1})] \\ &\leq (A+C+D)d(x_{n-1},x_n) + (B+D)d(x_n,x_{n+1}) \end{aligned}$$

This means that

$$(e-B-D)d(x_{n+1},x_n) \leq (A+C+D)d(x_{n-1},x_n)\ldots \quad (12.2)$$

Since $r(B+D) \leq r(B+D)+r(A+C+D) < 1$, then by proposition 2.1, $(e-B-D)$ is invertible. Furthermore,

$$(e-B-D)^{-1} = \sum_{i=0}^{\infty}(B+D)^i \ldots \quad (12.3)$$

Put $h = (e-B-D)^{-1}.(A+C+D)$. As $B+D$ commutes with $A+C+D$ it follows that

$$\begin{aligned}(e-B-D)^{-1}.(A+C+D) &= \left(\sum_{i=0}^{\infty}(B+D)^i\right)(A+C+D)\\ &= (A+C+D)\left(\sum_{i=0}^{\infty}(B+D)^i\right)\\ &= (A+C+D)\,(e-B-D)^{-1}.\end{aligned}$$

That is $(e-B-D)^{-1}$ commutes with $(A+C+D)$. Then by Lemma.6 and Lemma 7 that, we obtain

$$\begin{aligned}r(h) &= r[\,(e-B-D)^{-1}.(A+C+D)]\\ &\leq r[(e-B-D)^{-1}].\ r(A+C+D)\\ &\leq [1-(B+D)]^{-1}.r(A+C+D)\\ &= \frac{1}{1-(B+D)}.r(A+C+D)\\ &< 1\end{aligned}$$

Which illustrates that $(e-h)^{-1} = \sum_{i=0}^{\infty}(h)^i$ and $\|h^n\| \to 0 (n \to \infty)$. Thus for, we get

$$\begin{aligned}d(x_{n+1},x_n) &\leq (e-B-D)^{-1}.(A+C+D)\,d(x_n,x_{n-1})\\ &= h\,d(x_n,x_{n-1})\\ &\leq h^2 d(x_{n-1},x_{n-2})\\ &\leq \cdots \ldots . \leq h^n d(x_0,x_1)\end{aligned}$$

Now, Let $m > n \geq 1$. then it follows that

$$\begin{aligned}d(x_n,x_m) &\leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \ldots\ldots\ldots + d(x_{m-1},x_m)\\ &\leq (h^n + h^{n+1} + \cdots \ldots \ldots \ldots \ldots \ldots + h^{m-1})d(x_0,x_1)\\ &\leq (e + h + h^{n+1} + \cdots \ldots \ldots \ldots \ldots \ldots + h^{m-n-1})d(x_0,x_1)\\ &\leq \frac{h^n}{1-h}\,d(x_0,x_1)\\ &\leq (e-h)^{-1}h^n\,d(x_0,x_1)\ldots\ldots \quad (12.4)\end{aligned}$$

Since P is normal with normal constant K, and note that $\|h^n\| \to 0 (n \to \infty)$, we have

$$\|d(x_n,x_m)\| \leq K[\|(e-h)^{-1}h^n\,d(x_0,x_1)\|]$$

$$\leq K[\|(e-h)^{-1}\| \, \|h^n\| \, \|d(x_{0,}x_1)\|] \to 0(n \to \infty).$$

Hence $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*(n \to \infty)$, to verify $Sx^* = x^*$, we have

$$d(Sx^*, x^*) \leq d(Sx^*, Sx_n) + d(Sx_n, x^*)$$
$$\leq Ad(x^*, x_n) + Bd(Sx^*, x^*) + Cd(Sx_n, x^*)$$
$$+D[d(Sx^*, x_n) + d(Sx_n, x^*) \quad + d(x_{n+1,}u)$$
$$\leq Ad(x^*, x_{n,}) + Bd(Sx^*, x^*) + Cd(x_{n+1}, x^*)$$
$$+D[d(Sx^*, x^*) + d(x^*, x_n) + d(x_{n+1,}x^*] + d(x_{n+1,}x^*)$$

Implies that,

$(e - B - D)d(Sx^*, x^*) \leq (A + D)d(x^*, x_n) + (1+C + D)d(x_{n+1}, x^*)$

Note that, $(e - B - D)$ is invertible, then by the normality of , we have

$$\|d(Sx^*, x^*)\| \leq K[\{\|e-(B+D)\|\}^{-1}\|(A+D)\|\|d(x^*, x_n)\|$$
$$+\|(e + C + D)\|\|d(x_{n+1}, x^*)\|] \to 0(n \to \infty).$$

Hence, $\|d(Sx^*, x^*)\| = 0$.This implies $Sx^* = x^*$. So, x^* is a fixed point of S in X.

Now, y^* is another fixed point of S in X.

Then

$$d(x^*, y^*) \leq d(Sx^*, Sy^*)$$
$$\leq Ad(x^*, y^*) + B(Sx^*, x^*) + Cd(Sy^*, y^*) + d[d(Sx^*, y^*) + d(Sy^*, x^*)]$$
$$= (A + 2D)\, d(x^*, y^*)$$

Implies that $(e - A - 2D)\, d(x^*, y^*) \leq 0$.Now, by normality of P

Thus $\|d(x^*, y^*)\| \leq K\|(A + 2D)\| \, \|d(x^*, y^*)\| \to 0(n \to \infty)$. =>$\|d(x^*, y^*)\| = 0$

Implies that, $(x^*, y^*) = 0 =>x^* = y^*$. Thus$x^*$ is an unique fixed point of S in X. This theorem is proved.

Theorem 13: Let (X, d)be a complete cone metric spaces over Banach algebra A, P be a normal cone with a normal constant$K$. Suppose the mapping $S: X \to X$ satisfies the generalized Lipschitze conditions

$$\boldsymbol{d}(Sx, Sy) \leq Ad(x, y) + B[d(Sx, x) + d(Sy, y)]$$
$$+C[d(Sx, y) + d(Sy, x)] \ldots. \qquad (13.1)$$

for all $x, y \in X$, where $A, B, C, \in P$ are generalized Lipschitz constants with $r(B + C)$ $+r(A + B + C) < 1$. If $r(B + D)$commutes with$A + B + C$, then S kept a unique fixed point in X. And for any $x_0 \in X$, iterative sequence $\{S^n x\}$ converges to the fixed point.

Proof: Let x_0 is an arbitrary point in X and set$x_n = Sx_{n-1} = S^n x_0, n \geq 1.$

Put $x = x_n$ and $y = x_{n-1}$ from (3.4.1), we have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq d(Sx_{n,}Sx_{n-1}) \\ &\leq Ad(x_n, x_{n-1}) + B[d(Sx_n, x_n) + (Sx_{n-1}, x_{n-1})] \\ &+C[d(Sx_n, x_{n-1}) \ +d(Sx_{n-1}, x_n)] \\ &\leq Ad(x_n, x_{n-1}) + B[d(x_{n+1}, x_n) + \lambda_3 d(x_n, x_{n-1})] \\ &+D[d(x_{n+1}, x_{n-1}) + d(x_n, x_n)] \\ &\leq Ad(x_n, x_{n-1}) + B[d(x_{n+1}, x_n) + d(x_n, x_{n-1})] \\ &+C[d(x_n, x_{n-1}) + d(x_n, x_{n+1})] \\ &\leq (A + B + C)d(x_{n-1}, x_n) + (B + C)d(x_n, x_{n+1}) \end{aligned}$$

This means that

$$(e - B - C)\, d(x_{n+1}, x_n) \leq (A + B + C)d(x_{n-1}, x_n) \qquad (13.2)$$

Since $r(B + C) \leq r(B + C) + r(A + B + C) < 1$, then by proposition 2.1, $(e - B - C)$ is invertible. Furthermore,

$$(e - B - C)^{-1} = \sum_{i=0}^{\infty}(B + C)^i \qquad (13.3)$$

Put $h = (e - B - C)^{-1}.(A + B + C)$. As $B + D$commutes with $A + B + C$ it follows that

$$\begin{aligned} (e - B - C)^{-1}.(A + B + C) &= \left(\sum_{i=0}^{\infty}(B + C)^i\right)(A + B + C) \\ &= (A + B + C)\left(\sum_{i=0}^{\infty}(B + C)^i\right) \\ &= (A + B + C)\,(e - B - C)^{-1}. \end{aligned}$$

That is $(e - B - C)^{-1}$ commutes with $(A + B + C)$. Then by Lemma.6 and Lemma.7 that, we obtain

$$\begin{aligned} r(h) &= r[\,(e - B - C)^{-1}.(A + B + C)] \\ &\leq r[(e - B - C)^{-1}].\ r(A + B + C) \\ &\leq [1 - (B + C)]^{-1}.r(A + B + C) \\ &= \frac{1}{1-(B+C)}.r(A + B + C) \\ &< 1 \end{aligned}$$

Which illustrates that $(e - h)^{-1} = \sum_{i=0}^{\infty}(h)^i$ and $\|h^n\| \to 0 (n \to \infty)$. Thus for, we get

$$\begin{aligned} d(x_{n+1}, x_n) &\leq (e - B - D)^{-1}.(A + C + D)\, d(x_n, x_{n-1}) \\ &= h\, d(x_n, x_{n-1}) \\ &\leq h^2 d(x_{n-1}, x_{n-2}) \end{aligned}$$

$$\leq \cdots \ldots . \leq h^n d(x_0, x_1)$$

Now, Let $m > n \geq 1.$ then it follows that

$$d(x_{n,} x_m) \leq d(x_{n,} x_{n+1}) + d(x_{n+1,} x_{n+2}) + \ldots\ldots\ldots + d(x_{m-1,} x_m)$$

$$\leq (h^n + h^{n+1} + \cdots \ldots \ldots \ldots \ldots \ldots + h^{m-1}) d(x_{0,} x_1)$$

$$\leq (e + h + h^{n+1} + \cdots \ldots \ldots \ldots \ldots \ldots + h^{m-n-1}) d(x_{0,} x_1)$$

$$\leq \frac{h^n}{1-h} d(x_{0,} x_1)$$

$$\leq (e-h)^{-1} h^n \, d(x_{0,} x_1) \qquad (13.4)$$

Since P is normal with normal constant$K$,and note that $\|h^n\| \rightarrow 0 (n \rightarrow \infty)$, we have

$$\|d(x_n, x_m)\| \leq K[\|(e-h)^{-1} h^n \, d(x_{0,} x_1)\|]$$

$$\leq K[\|(e-h)^{-1}\| \, \|h^n\| \, \|d(x_{0,} x_1)\|] \rightarrow 0 (n \rightarrow \infty).$$

Hence $\{x_n\}$ is a Cauchy sequence in$X$. Since X is complete, there exists $x^* \in X$ such that $x_n \rightarrow x^* (n \rightarrow \infty)$. To verify$Sx^* = x^*$, we have

$$d(Sx^*, x^*) \leq d(Sx^*, Sx_n) + d(Sx_n, x^*)$$

$$\leq Ad(x^*, x_n) + B[d(Sx^*, x^*) + d(Sx_n, x^*)]$$

$$+C[d(Sx^*, x_n) + d(Sx_n, x^*)] + d(x_{n+1,} u).$$

$$\leq Ad(x^*, x_{n,}) + B[d(Sx^*, x^*) + d(x_{n+1}, x^*)]$$

$$+C[d(Sx^*, x^*) + d(x^*, x_n) + d(x_{n+1,} x^*] + d(x_{n+1,} x^*)$$

Implies that,

$$(e - B - C) d(Sx^*, x^*) \leq (A + C) d(x^*, x_n) + (1 + B + C) d(x_{n+1}, x^*)$$

Note that, $(e - B - C)$ is invertible, then by the normality of, we have

$$\|d(Sx^*, x^*)\| \leq K[\{\|e - (B + C)\|\}^{-1} \|(A + D)\| \|d(x^*, x_n)\|$$

$$+\|(e + B + C)\| \|d(x_{n+1}, x^*)\|] \rightarrow 0 (n \rightarrow \infty).$$

Hence, $\|d(Sx^*, x^*)\| = 0$.This implies $Sx^* = x^*$. So, x^* is a fixed point of S in X.

Now, y^* is another fixed point of S in X. Then

$$d(x^*, y^*) \leq d(Sx^*, Sy^*)$$

$$\leq Ad(x^*, y^*) + B[(Sx^*, x^*) + d(Sy^*, y^*)]$$

$$+C[d(Sx^*, y^*) + d(Sy^*, x^*)]$$

$$= (A + 2C) \, d(x^*, y^*).$$

Implies that $(e - A - 2C) \, d(x^*, y^*) \leq 0$.Now, by normality of P

Thus $\|d(x^*, y^*)\| \leq K \|(A + 2C)\| \, \|d(x^*, y^*)\| \rightarrow 0 (n \rightarrow \infty)$.

$=> \|d(x^*, y^*)\| = 0$

Implies that, $(x^*, y^*) = 0 => x^* = y^*$. Thus x^* is an unique fixed point of S in X. This theorem is proved.

Theorem 14: Let (X, d)be a complete cone metric spaces over Banach algebra A, P be a normal cone with a normal constant$K$. Suppose the mapping $S: X \to X$ satisfies the generalized Lipschitze conditions

$$\boldsymbol{d}(Sx, Sy) \leq a_1 d(x, y) + a_2[d(x, Sx) + d(y, Sy)] + a_3[d(x, Sy) + d(y, Sx)]$$
$$+ a_4[d(x, Sx) + d(x, y)] + a_5[d(y, Sy) + d(x, y)] \ldots \quad (14.1)$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4, a_5 are generalized Lipschitz constants with $r(a_2 + a_3 + a_5) \ + r\ (a_1 + a_2 + a_3 + 2a_4 + a_5) < 1$,If $(a_2 + a_3 + a_5)$commutes with$a_1 + a_2 + a_3 + 2a_4 + a_5$, then S has a unique fixed point in X. And for any $x_0 \in X$, iterative sequence $\{S^n x\}$ converges to the fixed point.

Proof: Choose $x_0 \in X$. Set $x_1 = Sx_{0,}, \ x_2 = Sx_1 = S^2 x_0 \ldots \ldots . x_{n+1} = Sx_n = S^n x_0$

Then we have,

$$d(x_n, x_{n+1}) \leq d(Sx_{n-1,} Sx_n) \quad \ldots\ldots \quad (14.2)$$
$$\leq a_1 d(x_{n-1}, x_n) + a_2[d(x_{n-1}, Tx_{n-1}) + dq(x_n, Tx_n)]$$
$$+ \ a_3[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})$$
$$+ a_4[d(x_{n-1}, Tx_{n-1}) + d(x_{n-1}, x_n)$$
$$+ a_5[q(x_n, Tx_n) + q(x_n, x_n)]$$
$$= a_1 d(x_{n-1}, x_n) + a_2[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$$
$$+ a_3[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)]$$
$$+ a_4[d(x_{n-1}, x_n) + d(x_{n-1}, x_n)]$$
$$+ a_5[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)]$$
$$d(x_n, x_{n+1}) \leq (a_1 + a_2 + a_3 + 2a_4 + a_5) d(x_{n-1}, x_n)$$
$$+ (a_2 + a_3 + a_5) q(x_n, x_{n+1})$$

Which means that

$$(e - a_2 - a_3 - a_5) d(x_n, x_{n+1}) \leq (a_1 + a_2 + a_3 + 2a_4 + a_5) d(x_{n-1}, x_n)$$

Since $r(a_2 + a_3 + a_5) \ + r\ (a_1 + a_2 + a_3 + 2a_4 + a_5) < 1$, then by proposition 2.1, $(e - a_2 - a_3 - a_5)$ is invertible. Furthermore,

$$(e - a_2 - a_3 - a_5)^{-1} = \sum_{i=0}^{\infty} (e - a_2 - a_3 - a_5)^i \ \ldots\ldots\ldots \ (3.5.3)$$

Put $h = (e - a_2 - a_3 - a_5)^{-1}.(a_1 + a_2 + a_3 + 2a_4 + a_5)$.As$(a_2 + a_3 + a_5)$ccommutes with$a_1 + a_2 + a_3 + 2a_4 + a_5$ it follows that

$$(e - a_2 - a_3 - a_5)^{-1}.(a_1 + a_2 + a_3 + 2a_4 + a_5)$$
$$= \left(\sum_{i=0}^{\infty}(e - a_2 - a_3 - a_5)^i\right)(a_1 + a_2 + a_3 + 2a_4 + a_5).$$
$$= \left((a_1 + a_2 + a_3 + 2a_4 + a_5).\right)\left(\sum_{i=0}^{\infty}(e - a_2 - a_3 - a_5)^i\right)$$
$$= (a_1 + a_2 + a_3 + 2a_4 + a_5)(e - a_2 - a_3 - a_5)^{-1}.$$

That is $(e - a_2 - a_3 - a_5)^{-1}$. commutes with $(a_1 + a_2 + a_3 + 2a_4 + a_5)$. Then by Lemma 2.6 and Lemma 2.7 that, we obtain

$$r(h) = r[(e - a_2 - a_3 - a_5)^{-1}.(a_1 + a_2 + a_3 + 2a_4 + a_5)]$$
$$\leq r[(e - a_2 - a_3 - a_5)^{-1}].\, r(a_1 + a_2 + a_3 + 2a_4 + a_5)$$
$$\leq [1 - (a_2 + a_3 + a_5)]^{-1}.r(a_1 + a_2 + a_3 + 2a_4 + a_5)$$
$$= \frac{1}{1-(a_2+a_3+a_5)}.r(a_1 + a_2 + a_3 + 2a_4 + a_5)$$
$$< 1$$

Which illustrates that $(e - h)^{-1} = \sum_{i=0}^{\infty}(h)^i$ and $\|h^n\| \to 0 (n \to \infty)$. Thus for, we get

$$d(x_{n+1}, x_n) \leq (e - a_2 - a_3 - a_5)^{-1}.(a_1 + a_2 + a_3 + 2a_4 + a_5)\, d(x_n, x_{n-1})$$
$$= h\, d(x_n, x_{n-1}) \leq h^2 d(x_{n-1}, x_{n-2})$$
$$\leq \cdots \ldots . \leq h^n d(x_0, x_1) \ldots .. \qquad (14.3)$$

Let $m > n \geq 1$. Then it follows that

Now, Let $m > n \geq 1$. then it follows that

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots\ldots\ldots + d(x_{m-1}, x_m)$$
$$\leq (h^n + h^{n+1} + \cdots \ldots \ldots \ldots \ldots \ldots + h^{m-1}) d(x_0, x_1)$$
$$\leq (e + h + h^{n+1} + \cdots \ldots \ldots \ldots \ldots \ldots + h^{m-n-1}) d(x_0, x_1)$$
$$\leq \frac{h^n}{1-h} d(x_0, x_1)$$
$$\leq (e - h)^{-1} h^n\, d(x_0, x_1) \ldots \ldots \qquad (14.4)$$

Since P is normal with normal constant$K$,and note that $\|h^n\| \to 0 (n \to \infty)$, we have

$$\|d(x_n, x_m)\| \leq K[\|(e - h)^{-1} h^n\, d(x_0, x_1)\|]$$
$$\leq K[\|(e - h)^{-1}\|\, \|h^n\|\, \|d(x_0, x_1)\|] \to 0 (n \to \infty).$$

Hence $\{x_n\}$ is a Cauchy sequence in$X$. Since X is complete, there exists $v \in X$ such that $x_n \to v (n \to \infty)$. To verify$Sv = v$ we have

Then we have

$$d(Sv,v) \leq d(Sx_n,Sv) + d(Sx_n,v)$$
$$\leq a_1 d(x_n,v) + a_2[d(x_n,Sx_n) + d(v,Sv)] + a_3[d(x_n,Sv) + d(v,Sx_n)]$$
$$+ a_4[d(x_n,Sx_n) + d(x_n,v)] + a_5[(v,Sv) + d(x_n,v)] + d(x_{n+1,}u),$$
$$\leq a_1 d(v,x_n) + a_2[d(x_n,x_{n+1}) + d(v,Sv)]$$
$$+a_3[d(v,Sv) + d(v,x_n) + d(v,x_{n+1})]$$
$$+ a_4[d(x_n,x_{n+1}) + d(v,x_n)] + a_5[d(v,Sv) + d(x_n,v)] + d(x_{n+1,}v),$$

This implies that,

$$(e - a_2 - a_3 - a_5)d(Sv,v) \leq (a_1 + a_3 + a_4 + a_5)d(x_{n,}v)$$
$$+(a_2 + a_4)d(x_n,x_{n+1}) + (1 + a_3)d(x_{n+1,}v)$$

Note that, $(e - a_2 - a_3 - a_5)$ is invertible, then by the normality of cone, we have

$$\|d(Sv,v)\| \leq K[\{\|e - (a_2 + a_3 + a_5)\|\}^{-1}\|a_1 + a_3 + a_4 + a_5)\|\|dd(x_{n,}v)\|$$
$$+\|((a_2 + a_4))\|\|d(x_n,x_{n+1})\| + \|(e + a_3)\|\|d(x_{n+1,}v)\|] \to 0(n \to \infty).$$

Hence, $\|d(Sv,v)\| = 0$.This implies $Sv = v$. So, v is a fixed point of S in X.

Next we prove that the uniqueness of the fixed point. Suppose that, there is another fixed point of w of S, and then we have

$$d(v,w) \leq d(Sv,Sw)$$
$$\leq a_1 d(v,w) + a_2[d(v,Sv) + d(w,Sw) + a_3[d(v,Sw) + d(w,Sv)]$$
$$+ a_4[d(v,Sv) + d(v,w)] + a_5[d(w,Sw) + d(v,w)]$$
$$= (a_1 + 2a_3 + a_4 + a_5)d(v,w).$$
$$\leq (a_1 + a_2 + 2a_3 + a_4 + a_5)\, d(v,w). \ldots \quad (14.6)$$

Since $r(a_2 + a_3 + a_5) + r\,(a_1 + a_2 + a_3 + 2a_4 + a_5) < 1$, then by normality of P.

Thus $\|d(x^*,y^*))\| \leq K\|\,(a_1 + a_2 + 2a_3 + a_4 + a_5)\|\,\|d(v,w)\| \to 0(n \to \infty)$.

$\Rightarrow \|d(v,w)\| = 0$

Implies that, $(v,w) = 0 \Rightarrow v = w$. Thus$v$ is an unique fixed point of S in X. This theorem is proved.

REFERENCES

Banach S. (1922). Sure less operations dans les ensembles abstraits et leur application aux equations integrals, Found. Math.3 (1), 133-181. http://eudml.org/doc/213289

Huang and Zhang. (2007). Cone metric spaces and fixed point theorems of contractive Mappings, J. Math. Anal. Appl. 332, 1468-1476. https://doi.org/10.1016/.jmaa.2005.03.087.

Rezapour, Sh. and Hamlbarani, R. (2008). Some notes on the paper-“Cone metric spaces and fixed point theorems of contractive mapping” J.of Math. Anal. Appl.345(2), 719-724. doi:10.1016/j.jmaa.2008.04.049.

Liu,H. and Xu,S.. (2013). Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mapping, Fixed point theory and applications, 2013:320. https://doi.org/10.1186/1687-1812-2013-320.

Shaoyuan Xu and Stojan Radenovic. (2014). Fixed point theorems of generalized Lipschitz mapping on cone metric spaces over Banach algebras without assumption of normality, Fixed point theory and applications, 2014:102. https://doi.org/10.1186/1687-1812-2014-102.

Huaping Huang and Stojan Rdenovics. (2015). Common fixed point theorems of Generalized Lipschitz Mapping in cone metric spaces over Banach algebras, Applied mathematics and information Sciences, 9(6), 2983-2990. DOI:10.12785/amis/090626

Qi yan, Jiandong Yin and Tao Wang. (2016). Fixed point and common fixed point theorems on ordered cone metric spaces over Banach algebra, J. of nonlinear Sci. and appl. 9, 1581-1589. doi.org/10.22436/jnsa.009.04.15.

Cho, Seong-Hoon. (2018). Fixed point theorems in complete cone metric spaces over Banach algebra, Hindawi journal of function spaces, vol.2018, Article ID9395057, 8 pages. https://doi.org/10.1155/2018/9395057

Yan Han and Shaoyuan Xu. (2018). Some new theorems on c- distance without continuity in cone metric spaces over Banach algebras, Hindawij. of function Spaces, vol. 2018, Article Id7463435, 10 Pages. doi.org/10.1155/2018/7463435.

Huaping,H., Stojan Radenovic and Tatjana Dosenovic. (2015). Some common fixed point theorems on c- distance in cone meteic spaces over Banch algebras, 14(2), 180-193. https://www.researchgate.net/publication/317014722.

Ahmad,A. Mitrovic, Z.D.and Salinke, J.N. (2017) On some open problems in cone metric space over Banach algebra, J.of linear and Topological algebra, vol. 6(4), 261-267.

Sastri, K.P.R. Naidu,A.G. and Bakeshaie, T. (2012). Fixed point theorems in cone metric spaces with Banach algebra cones, Int. J. Math. Sci. and Engg. Appl. 6, 129-136.

Rudin, W. (1991).S Functional analysis, Mc Graw-Hill, New York,NY, USA,2nd edition.