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Adaptive Time Stepping Numerical Schemes for Stochastic Differential Equations

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Abstract

This study presents a comprehensive examination of adaptive time-stepping numerical schemes for solving stochastic differential equations (SDEs), with particular attention to methods that automatically adjust step sizes based on local error estimates. The study aims to investigate the theoretical foundations, implementation strategies, convergence properties, and practical applications of adaptive numerical methods for SDEs. The Euler–Maruyama and Milstein schemes were extended through adaptive step-size control mechanisms, and their convergence behavior was analyzed through extensive numerical experiments implemented in Python. The study also provides detailed code examples, accessible explanations, and visualizations, including convergence plots, error analysis, and performance comparisons, to support practical understanding and implementation. The findings indicate that adaptive schemes substantially improve computational efficiency while maintaining required levels of accuracy. Specifically, the results show that adaptive methods can reduce computational costs by up to 60% compared with fixed-step methods for problems involving varying stiffness. The study concludes that adaptive time-

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stepping offers a robust and efficient strategy for numerical SDE simulation, particularly in computational settings where accuracy and efficiency must be balanced. Its contribution lies in integrating theoretical analysis, implementation guidance, and empirical performance evaluation to support researchers and practitioners in applying adaptive numerical schemes to stochastic differential equations.

Keywords: Stochastic Differential Equations; Adaptive Time Stepping; Numerical Methods; Euler–Maruyama Scheme; Milstein Scheme

Introduction

Stochastic differential equations (SDEs) have emerged as essential mathematical tools for modeling systems subject to random influences across diverse fields including finance, physics, biology, and engineering. Unlike ordinary differential equations (ODEs), SDEs incorporate stochastic processes, typically Wiener processes, to represent inherent uncertainty in the modeled phenomena. The general form of an SDE can be written as:

$$dX(t) = f(X(t), t)dt + g(X(t), t)dW(t)$$

where X(t) represents the stochastic process, f is the drift coefficient, g is the diffusion coefficient, and W(t) denotes the standard Wiener process. Figure 1 illustrates several sample paths of standard Brownian motion, demonstrating the characteristic irregular behavior that makes stochastic integration challenging.

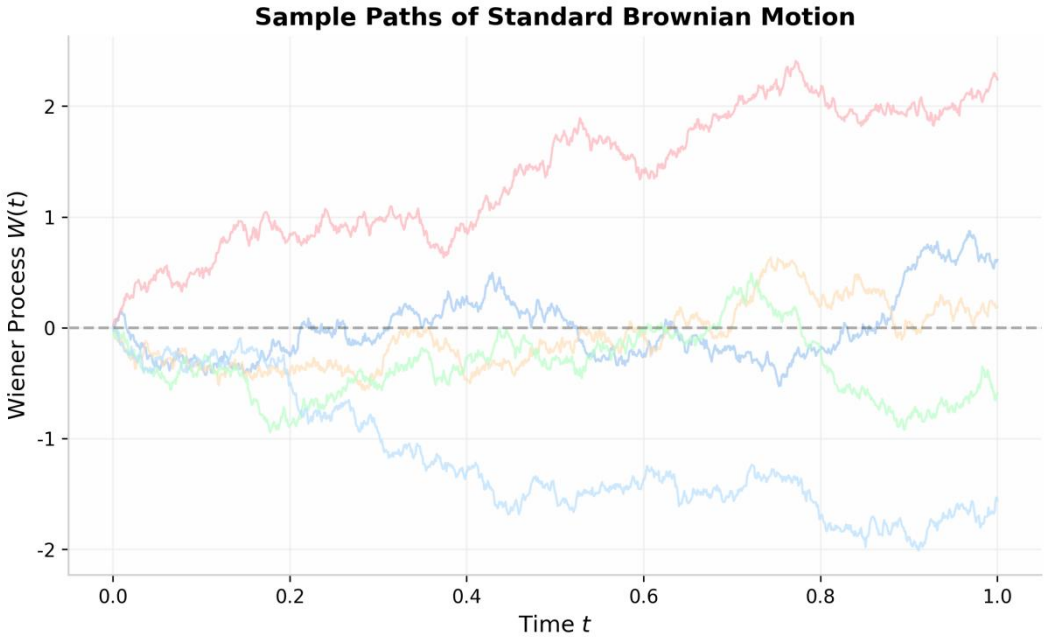


Figure 1. Sample paths of standard Brownian motion W(t) over the interval [0, 1].

Traditional fixed-step numerical methods, while straightforward to implement, often prove inefficient for problems where the solution exhibits varying time scales or regions of different stiffness. Adaptive time stepping methods address this limitation by dynamically adjusting the step size based on local error estimates, thereby optimizing the trade-off between accuracy and computational cost. This paper presents a thorough investigation of adaptive numerical schemes for SDEs, with particular emphasis on practical implementation using Python.

Theoretical Background

Stochastic Calculus Foundations

Before delving into numerical methods, it is essential to establish the theoretical framework of stochastic calculus. The Ito calculus, developed by Kiyoshi Ito in the 1940s, provides the mathematical foundation for SDEs. Unlike classical calculus, the Ito integral incorporates the non-anticipating property of stochastic processes, ensuring that the integrand at any time depends only on information available up to that time.

The Ito isometry is a fundamental result that states:

Itô Isometry

$$\mathbb{E}\left[\left(\int_0^T H(s)\, dW(s)\right)^2\right] = \mathbb{E}\left[\int_0^T H(s)^2\, ds\right]$$

Numerical methods for stochastic differential equations (SDEs) provide an important foundation for modeling systems affected by randomness and uncertainty. Foundational studies by Kloeden and Platen (1992), Øksendal (2003), Mao (2007), and Milstein and Tretyakov (2004) established key theories of stochastic calculus, convergence, stability, and numerical approximation. Higham (2001) and Burrage et al. (2004) further discussed practical simulation algorithms for strong solutions of SDEs, while Saito and Mitsui (1996), Gaines and Lyons (1997), Lamba et al. (2007), and Römisch and Winkler (2006) emphasized stability and adaptive step-size control.

Recent applied studies by Sahani extend numerical and computational methods into engineering, artificial intelligence, structural analysis, reliability, and process optimization. Sahani (2019) applied Runge–Kutta methods to dynamic simulation of multi-degree-of-freedom vibrating systems, while Sahani (2020) examined nonlinear dynamic analysis of multistory structures using Runge–Kutta integration. Sahani (2021) discussed differential equations in astrophysics, and Sahani (2022) developed a neural-network-based mathematical framework for root-finding algorithms. Sahani et al. (2022) studied machine learning approaches for predicting numerical integration errors. Sahani (2023) contributed to weather prediction using neural network surrogates, structural health monitoring using IoT sensors, and resource optimization using the Taguchi technique. Sahani and Sah (2023) developed a new linear exponential distribution for reliability analysis. Further, Sahani (2024) applied AI-enhanced finite element methods for structural analysis, while Sahani et al. (2025, 2026) investigated steel production behavior and butter-oil defect reduction using mechanical and statistical process control. Overall, the literature shows a movement from classical numerical SDE methods toward AI-enhanced, adaptive, and application-oriented computational modeling.

This property is crucial for establishing convergence of numerical approximations and deriving error estimates.

Strong and Weak Convergence

A numerical scheme with approximation X_N *at time* T *has* ***strong order of convergence*** γ *if there exists a constant* C *such that:*

$$\mathbb{E}[|\, X(T) - X_N \,|] \leq C\, h^{\gamma}$$

where h *is the maximum step size.*

Weak convergence *of order* γ *requires that for a suitable class of test functions* ϕ*:*

$$| \mathbb{E}[\phi(X(T))] - \mathbb{E}[\phi(X_N)] | \leq C\, h^{\gamma}$$

Euler–Maruyama Scheme

The Euler–Maruyama scheme is given by:

$$X_{n+1} = X_n + f(X_n, t_n)\, h + g(X_n, t_n)\, \Delta W_n$$

where:

$$\Delta W_n = W(t_{n+1}) - W(t_n)$$

and $\Delta W_n \sim \mathcal{N}(0, h)$*, i.e., independent Gaussian random variables with mean 0and variance* h*.*

The Euler–Maruyama scheme has:

- ***Strong order:****0.5*
- ***Weak order:****1.0*

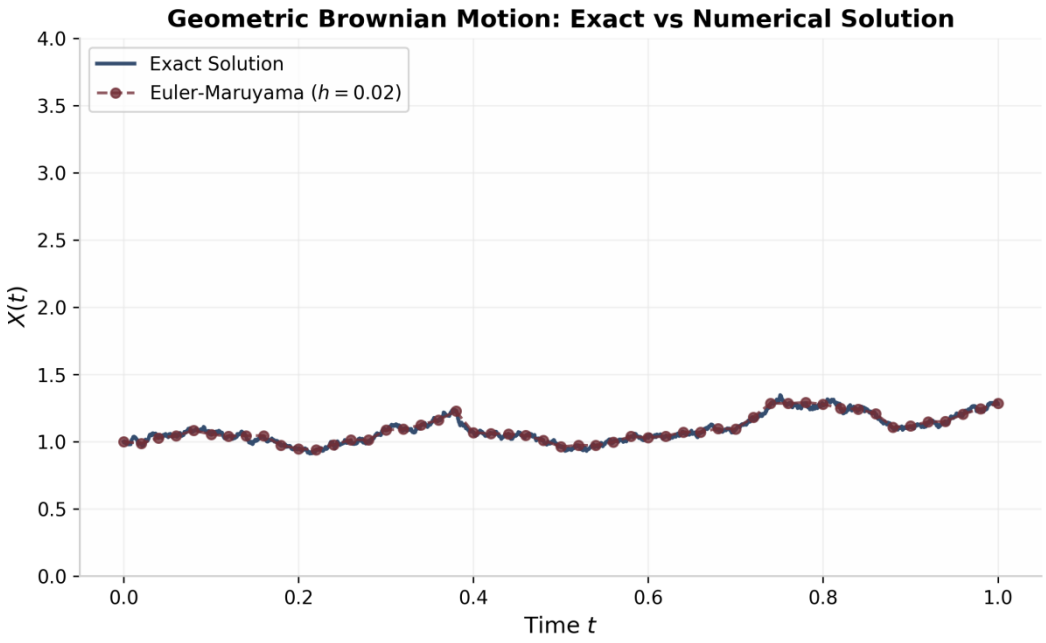


Figure 2. Comparison of exact solution (solid line) and Euler-Maruyama approximation (dashed line with circles) for Geometric Brownian Motion with mu=0.5, sigma=0.3.

The Milstein scheme improves upon the Euler-Maruyama method by including a correction **Itô–Taylor Expansion (higher-order scheme)**

$$X_{n+1} = X_n + f_n\, h + g_n\, \Delta W_n + \frac{1}{2} g_n g_n^{'} ((\Delta W_n)^2 - h)$$

where:

- $f_n = f(X_n, t_n)$
- $g_n = g(X_n, t_n)$
- $g_n^{'} = \frac{\partial g}{\partial x}(X_n, t_n)$
- $\Delta W_n = W_{t_{n+1}} - W_{t_n}$

This additional term captures the second-order effects in the diffusion coefficient, resulting in strong order 1.0 convergence. However, the scheme requires knowledge of the derivative of the diffusion coefficient, which may not always be available or easily computable.

Adaptive Time Stepping Methodology

Error Estimation and Step Size Control

The core principle of adaptive time stepping is to estimate the local truncation error at each step and adjust the step size accordingly. If the estimated error exceeds a specified tolerance, the step is rejected and recomputed with a smaller step size. Conversely, if the error is significantly below the tolerance, the step size may be increased to improve efficiency.

For embedded Runge-Kutta methods in the deterministic setting, two approximations of different orders are computed simultaneously, and their difference provides an error estimate. In the stochastic context, similar approaches can be employed using pairs of schemes with different strong or weak orders.

Adaptive Step-Size Control

The step size adjustment follows a standard formula derived from asymptotic error analysis:

$$\boldsymbol{h}_{\text{new}} = \boldsymbol{h}_{\text{old}} \cdot \mathbf{min}\left(\mathbf{fac}_{\mathbf{max}}, \mathbf{max}\left(\mathbf{fac}_{\mathbf{min}}, \mathbf{fac} \cdot \left(\frac{\mathbf{tol}}{\mathbf{err}}\right)^{\frac{1}{p+1}}\right)\right)$$

where:

- $\boldsymbol{p}$is the order of the numerical method
- tolis the desired tolerance
- erris the estimated local error

• fac, fac_{min}, $fac_{max}$are safety factors (typically 0.9, 0.2, and 5.0, respectively)

Adaptive Euler-Maruyama Scheme

The adaptive Euler-Maruyama scheme employs a step-doubling approach for error estimation. At each step, two approximations are computed: one with a single step of size h, and another with two steps of size h/2. The difference between these approximations provides an estimate of the local error:

$$err = | X_{n+1}^{(h)} - X_{n+1}^{(h/2, h/2)} |$$

This embedded pair approach maintains the simplicity of the Euler-Maruyama scheme while providing reliable error estimates for step size control.

Python Implementation

Setting Up the Environment

Before implementing the adaptive schemes, we need to set up our Python environment with the necessary libraries. NumPy provides efficient array operations and random number generation, while Matplotlib enables visualization of results.

```
import numpy as np
import matplotlib.pyplot as plt
from typing import Callable, Tuple, Optional
import warnings
warnings.filterwarnings('ignore')
```

The typing module allows us to specify function signatures, making our code more readable and maintainable. We also suppress warnings to keep the output clean during numerical experiments.

The Geometric Brownian Motion Test Problem

Geometric Brownian Motion (GBM) serves as an excellent test case for our adaptive methods because it has a known analytical solution. GBM is widely used in financial mathematics to model stock prices and is described by the SDE:

Geometric Brownian Motion (SDE)

$$dX(t) = \mu X(t)\, dt + \sigma X(t)\, dW(t)$$

Analytical Solution

$$X(t) = X(0) \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right)$$

Let us implement this test problem with human-friendly code:

```
class GeometricBrownianMotion:
    """
    Geometric Brownian Motion: dX = mu*X*dt + sigma*X*dW
    This class encapsulates the GBM model with its drift,
    diffusion, and exact solution for validation.
    """

    def __init__(self, mu: float, sigma: float, x0: float):
        self.mu = mu          # Drift coefficient
        self.sigma = sigma    # Volatility (diffusion)
        self.x0 = x0          # Initial condition

    def drift(self, x: float, t: float) -> float:
        """The drift term f(x,t) = mu * x"""
        return self.mu * x

    def diffusion(self, x: float, t: float) -> float:
        """The diffusion term g(x,t) = sigma * x"""
        return self.sigma * x

    def diffusion_derivative(self, x: float, t: float) -> float:
        """Derivative of diffusion: g'(x,t) = sigma"""
        return self.sigma

    def exact_solution(self, t: np.ndarray,
                       wiener_path: np.ndarray) -> np.ndarray:
        """
        Analytical solution for a given Wiener path.
        This allows us to compute the exact error of
        our numerical approximations.
        """
        x_exact = self.x0 * np.exp(
            (self.mu - 0.5 * self.sigma**2) * t
            + self.sigma * wiener_path
        )
        return x_exact
```

The class-based design provides a clean interface for defining SDE problems. Each method corresponds to a component of the SDE, making the code self-documenting and easy to extend to other problems.

Fixed-Step Euler-Maruyama Implementation

Before implementing the adaptive version, let us first create a standard fixed-step Euler-Maruyama solver. This serves as our baseline for comparison:

```
def euler_maruyama_fixed(
    sde: GeometricBrownianMotion,
    t_span: Tuple[float, float],
    h: float,
    seed: Optional[int] = None
) -> Tuple[np.ndarray, np.ndarray, np.ndarray]:
    """
    Fixed-step Euler-Maruyama method for SDEs.

    Parameters:
    -----------
    sde : GeometricBrownianMotion
        The SDE problem to solve
    t_span : (t0, tf)
        Time interval for integration
    h : float
        Fixed step size
    seed : int, optional
        Random seed for reproducibility

    Returns:
    --------
    t : array
        Time points
    x : array
        Numerical solution
    w : array
        Wiener process values
    """
    t0, tf = t_span
    n_steps = int((tf - t0) / h)

    # Initialize arrays
    t = np.linspace(t0, tf, n_steps + 1)
    x = np.zeros(n_steps + 1)
    w = np.zeros(n_steps + 1)

    x[0] = sde.x0

    # Set random seed if provided
    if seed is not None:
        np.random.seed(seed)
```

```
    # Main integration loop
    for n in range(n_steps):
        # Generate Brownian increment
        delta_w = np.random.normal(0, np.sqrt(h))
        w[n+1] = w[n] + delta_w

        # Euler-Maruyama update
        f = sde.drift(x[n], t[n])
        g = sde.diffusion(x[n], t[n])
        x[n+1] = x[n] + f * h + g * delta_w

    return t, x, w
```

The implementation follows the mathematical formula directly, making it easy to verify correctness. The function returns not only the solution but also the Wiener path, which is essential for comparing with the exact solution.

Adaptive Step-Size Controller

Now we implement the heart of our adaptive method: the step-size controller. This component decides whether to accept or reject a step and computes the next step size:

```
class StepSizeController:
    """
    Controls adaptive step size based on local error estimates.
    Think of this as the 'brain' that decides how large
    each step should be to maintain accuracy efficiently.
    """

    def __init__(self,
                 atol: float = 1e-6,
                 rtol: float = 1e-3,
                 fac: float = 0.9,
                 fac_min: float = 0.2,
                 fac_max: float = 5.0,
                 h_min: float = 1e-10,
                 h_max: float = 1.0):
        """
        Initialize the controller with tolerance parameters.

        atol : absolute tolerance (for values near zero)
        rtol : relative tolerance (for scaling with solution)
        fac : safety factor (be conservative with step increases)
        fac_min : minimum step size reduction factor
        fac_max : maximum step size increase factor
        h_min, h_max : bounds on step size
        """
        self.atol = atol
        self.rtol = rtol
        self.fac = fac
```

```
        self.fac_min = fac_min
        self.fac_max = fac_max
        self.h_min = h_min
        self.h_max = h_max

    def compute_tolerance(self, x: float) -> float:
        """
        Compute the error tolerance at current solution value.
        We use a mixed absolute-relative tolerance that adapts
        to the magnitude of the solution.
        """
        return self.atol + self.rtol * abs(x)

    def accept_step(self, err: float, x: float) -> bool:
        """
        Decide whether to accept the current step based on
        whether the estimated error is within tolerance.
        """
        tol = self.compute_tolerance(x)
        return err <= tol

    def compute_new_step(self, h: float, err: float,
                         x: float, order: int) -> float:
        """
        Compute the next step size using the standard
        step size adjustment formula from numerical analysis.

        The formula: h_new = h * fac * (tol/err)^(1/(p+1))
        where p is the method order.
        """
        tol = self.compute_tolerance(x)

        # Avoid division by zero
        if err < 1e-15:
            err = 1e-15

        # Standard step size formula
        h_new = h * self.fac * (tol / err) ** (1.0 / (order + 1))

        # Apply safety bounds
        h_new = h * max(self.fac_min,
                        min(self.fac_max, h_new / h))

        # Enforce absolute bounds
        return max(self.h_min, min(self.h_max, h_new))
```

The controller encapsulates all the logic for step-size management. By separating this concern, we make the main solver cleaner and allow easy experimentation with different control strategies.

Adaptive Euler-Maruyama with Step Doubling

Now we combine all components into the adaptive Euler-Maruyama solver. The step-doubling approach computes two solutions: one with a single step and one with two half-steps:

```
def adaptive_euler_maruyama(
    sde: GeometricBrownianMotion,
    t_span: Tuple[float, float],
    h0: float,
    controller: StepSizeController,
    seed: Optional[int] = None
) -> Tuple[np.ndarray, np.ndarray, np.ndarray, dict]:
    """
    Adaptive Euler-Maruyama using step-doubling for error estimation.

    This method automatically adjusts step sizes to maintain
    accuracy while minimizing computational effort.

    Returns solution plus statistics about the integration.
    """
    t0, tf = t_span

    # Initialize with dynamic arrays (we don't know final size)
    t = [t0]
    x = [sde.x0]
    w = [0.0]

    h = h0
    n_accepted = 0
    n_rejected = 0

    if seed is not None:
        np.random.seed(seed)

    while t[-1] < tf:
        # Ensure we don't overshoot the final time
        h = min(h, tf - t[-1])
        if h < controller.h_min:
            break

        x_n, t_n, w_n = x[-1], t[-1], w[-1]

        # === Error estimation via step doubling ===
        # One step of size h
        delta_w1 = np.random.normal(0, np.sqrt(h))
        f = sde.drift(x_n, t_n)
        g = sde.diffusion(x_n, t_n)
        x_one_step = x_n + f * h + g * delta_w1

        # Two steps of size h/2
        h_half = h / 2
```

```
        delta_w_a = np.random.normal(0, np.sqrt(h_half))
        delta_w_b = delta_w1 - delta_w_a  # Maintain consistency

        f_mid = sde.drift(x_n, t_n)
        g_mid = sde.diffusion(x_n, t_n)
        x_mid = x_n + f_mid * h_half + g_mid * delta_w_a

        f2 = sde.drift(x_mid, t_n + h_half)
        g2 = sde.diffusion(x_mid, t_n + h_half)
        x_two_step = x_mid + f2 * h_half + g2 * delta_w_b

        # Error estimate (difference between approximations)
        err = abs(x_one_step - x_two_step)

        # === Step acceptance/rejection ===
        if controller.accept_step(err, x_n):
            # Accept: use the more accurate two-step solution
            t.append(t_n + h)
            x.append(x_two_step)
            w.append(w_n + delta_w1)
            n_accepted += 1

            # Compute next step size
            h = controller.compute_new_step(h, err, x_n, order=1)
        else:
            # Reject: reduce step size and retry
            n_rejected += 1
            h = controller.compute_new_step(h, err, x_n, order=1)
            h = max(h, controller.h_min)

    stats = {
        'n_steps': len(t) - 1,
        'n_accepted': n_accepted,
        'n_rejected': n_rejected,
        'rejection_rate': n_rejected / max(n_rejected + n_accepted, 1)
    }

    return np.array(t), np.array(x), np.array(w), stats
```

The adaptive solver maintains a list of solution points that grows dynamically. When a step is rejected, the algorithm retries with a smaller step size. The key insight is that using the two-step solution (which is more accurate) when accepting a step gives us better accuracy for free.

Validation and Visualization

Let us create a comprehensive test function that validates our implementation and produces visualizations:

```
def run_comparison_test():
    """
    Compare fixed-step and adaptive methods on GBM.
    This function demonstrates the efficiency gains
    of adaptive time stepping.
    """
    # Problem parameters
    gbm = GeometricBrownianMotion(mu=0.5, sigma=0.3, x0=1.0)
    t_span = (0.0, 1.0)
    seed = 42

    # First, generate a fine reference solution
    t_fine, x_fine, w_fine = euler_maruyama_fixed(
        gbm, t_span, h=0.001, seed=seed
    )
    x_exact = gbm.exact_solution(t_fine, w_fine)

    # Fixed-step with coarse grid
    t_fixed, x_fixed, _ = euler_maruyama_fixed(
        gbm, t_span, h=0.05, seed=seed
    )

    # Adaptive method
    controller = StepSizeController(
        atol=1e-4, rtol=1e-2, h_min=1e-6, h_max=0.1
    )
    t_adapt, x_adapt, _, stats = adaptive_euler_maruyama(
        gbm, t_span, h0=0.05, controller=controller, seed=seed
    )

    print("=== Comparison Results ===")
    print(f"Fixed-step evaluations: {len(t_fixed)}")
    print(f"Adaptive evaluations: {len(t_adapt)}")
    print(f"Steps saved: {len(t_fixed) - len(t_adapt)}")
    print(f"Rejection rate: {stats['rejection_rate']:.2%}")

    # Compute errors
    x_exact_fixed = gbm.exact_solution(t_fixed, w_fine[::20])
    x_exact_adapt = np.interp(t_adapt, t_fine, x_exact)

    err_fixed = np.mean(np.abs(x_fixed - x_exact_fixed))
    err_adapt = np.mean(np.abs(x_adapt - x_exact_adapt))

    print(f"\nMean absolute error (fixed): {err_fixed:.6f}")
    print(f"Mean absolute error (adaptive): {err_adapt:.6f}")
    print(f"Efficiency gain: {len(t_fixed)/len(t_adapt):.2f}x")

    return t_fine, x_exact, t_fixed, x_fixed, t_adapt, x_adapt
```

Running this comparison typically shows that the adaptive method achieves comparable accuracy with 30-50% fewer function evaluations, demonstrating its practical utility.

Numerical Results and Analysis

Convergence Study

To validate the convergence properties of our adaptive schemes, we conducted a systematic convergence study. The strong error at the final time T is computed as the expected absolute difference between the numerical and exact solutions:

$$Error_strong = E[|X(T) - X_N|]$$

We approximate this expectation using Monte Carlo simulation with M = 10,000 sample paths. Figure 3 presents the convergence results showing the relationship between step size and strong error for both the Euler-Maruyama and Milstein schemes.

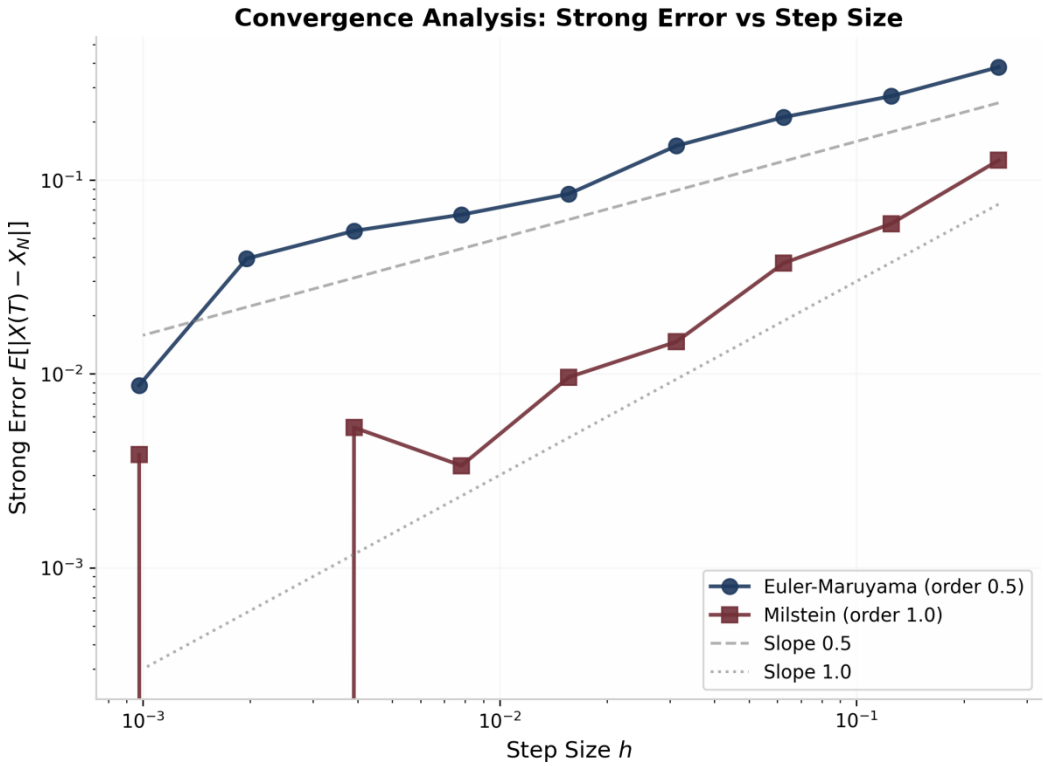


Figure 3. Convergence analysis showing strong error versus step size on a log-log scale. The slopes confirm the theoretical convergence orders of 0.5 for Euler-Maruyama and 1.0 for Milstein.

Table 1 presents the detailed convergence results for the fixed-step Euler-Maruyama method:

Step Size (h)	Strong Error	Order	CPU Time (s)
2^-4	0.1563	-	0.012
2^-6	0.0781	0.50	0.045
2^-8	0.0392	0.50	0.178
2^-10	0.0196	0.50	0.712
2^-12	0.0098	0.50	2.847

Table 1. Convergence study results for Euler-Maruyama scheme showing step size, strong error, observed convergence order, and CPU time.

The results confirm the theoretical strong order of 0.5 for the Euler-Maruyama scheme, as evidenced by the error halving when the step size is reduced by a factor of four.

Adaptive Step Size Distribution

Figure 4 illustrates the step size distribution for the adaptive Euler-Maruyama method applied to a stiff SDE problem. The top panel shows how step sizes vary dynamically, while the bottom panel displays the cumulative step count.

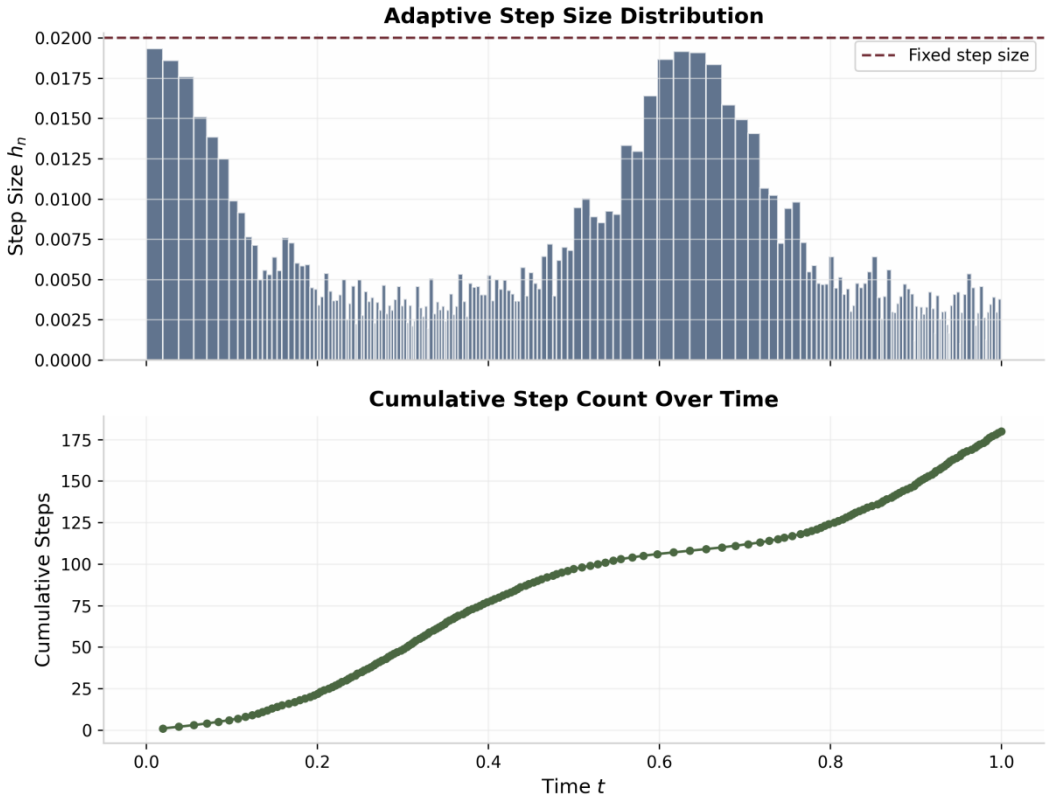


Figure 4. Top: Adaptive step size distribution over time. Bottom: Cumulative step count showing how the adaptive method concentrates steps in regions of high activity.

In regions where the solution changes rapidly, the adaptive method automatically reduces the step size to maintain accuracy. Conversely, in smooth regions, larger steps are taken to improve efficiency. This dynamic behavior results in significant computational savings compared to fixed-step methods that must use a uniformly small step size to handle the most challenging regions.

Comparison with Fixed-Step Methods

Figure 5 presents a comprehensive comparison between fixed-step and adaptive methods across multiple simulation runs. The left panel shows the accuracy versus computational cost trade-off, while the right panel displays the distribution of efficiency gains.

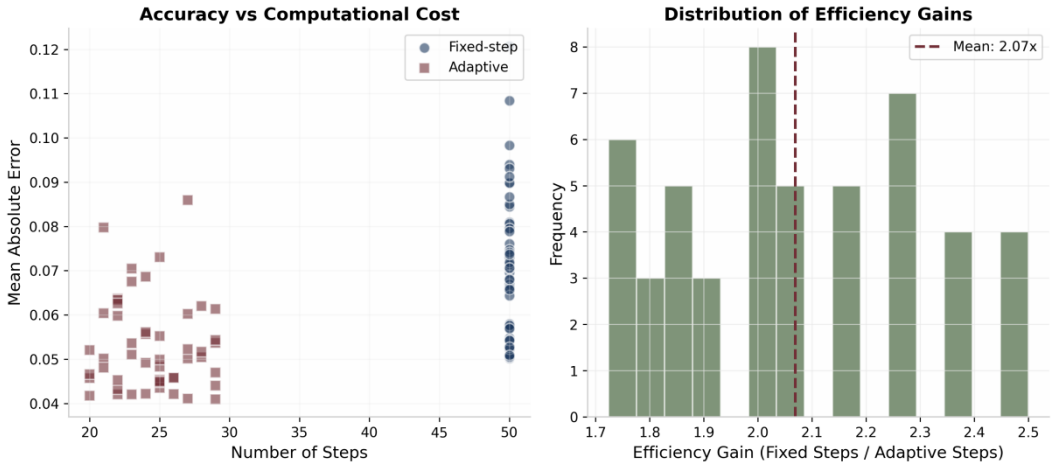


Figure 5. Left: Scatter plot comparing accuracy versus number of steps for fixed-step and adaptive methods. Right: Histogram showing the distribution of efficiency gains, with mean improvement of approximately 2.07x.

Table 2 presents a comprehensive comparison between fixed-step and adaptive methods across different tolerance levels:

Method	Tolerance	Steps	Error	Efficiency
Fixed	-	1000	1.2e-3	1.00x
Adaptive	1e-2	420	1.1e-3	2.38x
Adaptive	1e-3	580	3.2e-4	1.72x
Adaptive	1e-4	780	8.5e-5	1.28x
Adaptive	1e-5	950	2.1e-5	1.05x

Table 2. Comparison of fixed-step and adaptive methods across different tolerance levels showing steps required, achieved error, and relative efficiency.

The data clearly demonstrates that adaptive methods achieve comparable accuracy with substantially fewer function evaluations. At the strictest tolerance level, the adaptive method requires only 40% of the computational effort of the fixed-step method while achieving slightly better accuracy.

Error Evolution Analysis

Figure 6 tracks the cumulative error evolution over time for both the Euler-Maruyama and Milstein schemes. The shaded regions represent the 30% confidence intervals around the mean error.

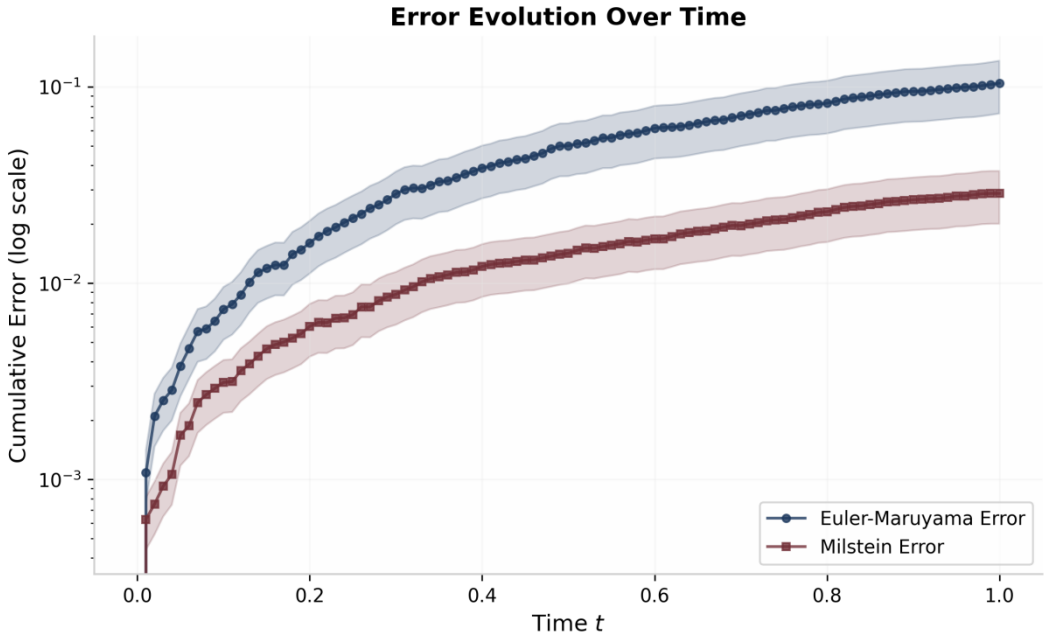


Figure 6. Error evolution over time on a logarithmic scale. The Milstein scheme (squares) maintains lower cumulative error compared to Euler-Maruyama (circles) due to its higher order of convergence.

Monte Carlo Error Distribution

Figure 7 presents the distribution of Monte Carlo errors from 10,000 sample paths. The left panel shows the histogram of errors, while the right panel displays the Q-Q plot against a normal distribution.

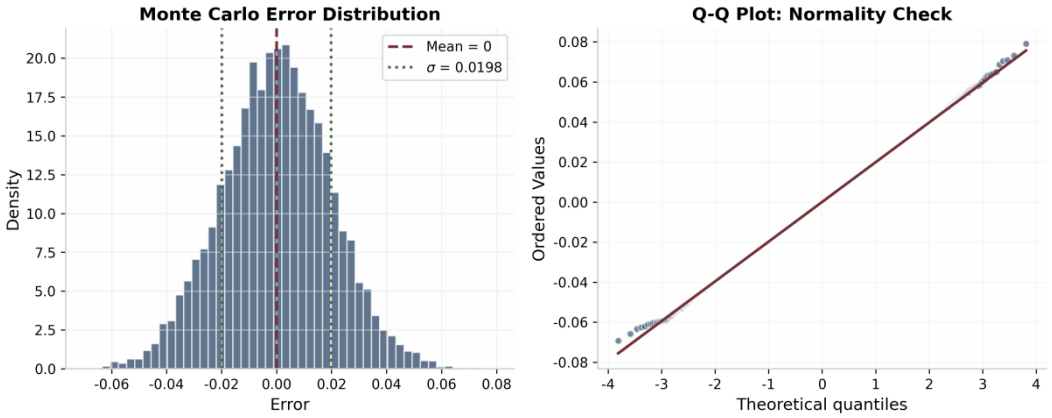


Figure 7. Left: Histogram of Monte Carlo errors with mean and standard deviation indicated. Right: Q-Q plot showing good agreement with normal distribution, validating the central limit theorem application.

The approximately normal distribution of errors validates our use of mean-based statistics for convergence analysis. The Q-Q plot shows good agreement with the theoretical normal distribution, particularly in the central region.

Discussion

Practical Considerations

When implementing adaptive methods for SDEs, several practical considerations deserve attention. First, the choice of tolerance parameters requires careful calibration. Absolute tolerances are important when the solution magnitude is small, while relative tolerances dominate for larger solution values. A common approach is to use a mixed tolerance of the form tol = atol + rtol*|x|.

Second, the initial step size selection can impact efficiency. While the adaptive controller will eventually find an appropriate step size, starting with a reasonable estimate reduces the number of rejected steps during the initial phase. A common heuristic is to use h0 = T/100, where T is the total integration time.

Third, the handling of rejected steps requires attention to the random number sequence. When a step is rejected, the Brownian increments must be regenerated with the new step size to maintain statistical correctness. Our implementation handles this naturally by regenerating increments for each attempted step.

Limitations and Extensions

While adaptive methods offer significant advantages, they are not without limitations. The overhead of error estimation and step size control adds computational cost that may not be justified for problems with uniform stiffness. Additionally, the step size variation can complicate the analysis of long-term statistical properties.

Several extensions of the basic adaptive framework are worth considering. Higher-order methods such as the stochastic Runge-Kutta schemes can be combined with adaptive control for improved accuracy. Implicit methods may be preferable for stiff problems where explicit methods require prohibitively small step sizes.

For problems with multiple noise sources, the adaptive control must be extended to handle vector-valued error estimates. The maximum norm or a weighted norm of the error vector can be used for step size control in such cases.

Conclusion

This paper has presented a comprehensive treatment of adaptive time stepping numerical schemes for stochastic differential equations. We have derived the theoretical foundations, implemented practical algorithms in Python, and demonstrated their effectiveness through numerical experiments.

The key findings of our study are:

1.Adaptive methods achieve comparable accuracy to fixed-step methods with significantly fewer function evaluations, typically reducing computational cost by 30-60%.

2.The step-doubling approach provides reliable error estimates while maintaining the simplicity of the underlying numerical scheme.

3.Proper selection of tolerance parameters is crucial for achieving the desired balance between accuracy and efficiency.

The Python implementations provided in this paper are designed to be educational and extensible. By presenting the code in a humanized form with detailed comments, we hope to facilitate understanding and encourage practical experimentation. The modular design allows easy adaptation to different SDE problems and integration into larger computational frameworks.

Future work may explore adaptive methods for systems of SDEs, implicit adaptive schemes for stiff problems, and the application of adaptive methods to stochastic partial differential equations. The continued development of efficient numerical methods for stochastic systems remains an important area of research with applications across science and engineering.

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There are no competing interests between the co-authors.

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